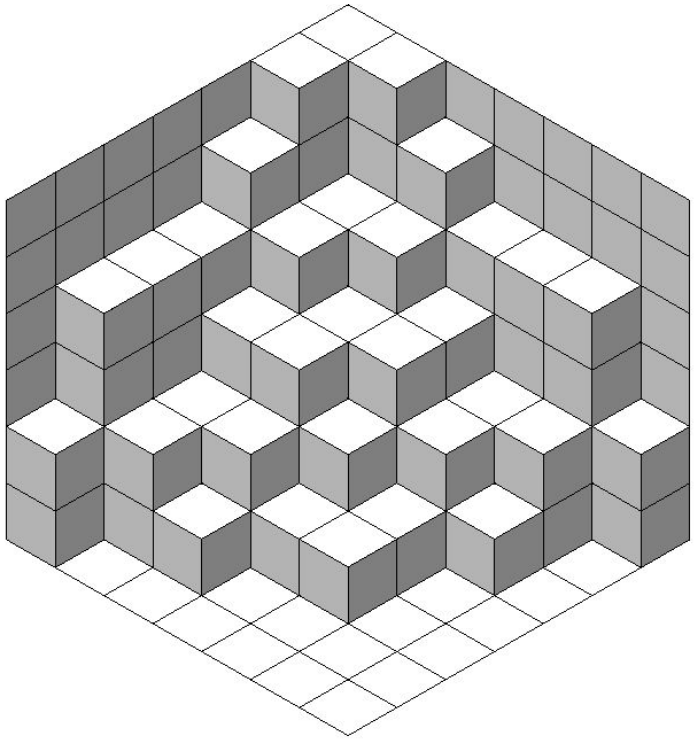
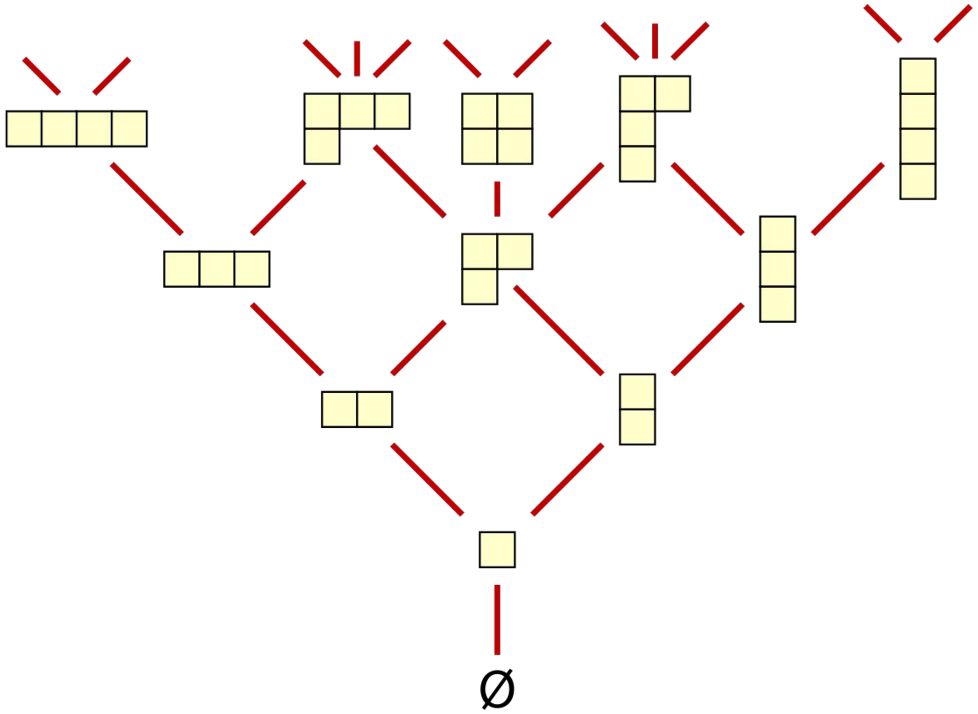
Berkeley Math Circle

Partitions



Peter Koroteev
UC Berkeley

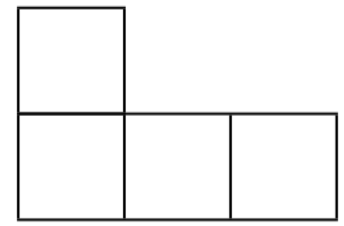


Partitions

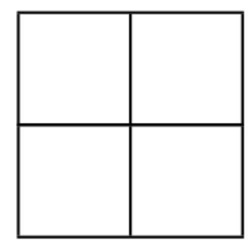
There are several ways to decompose an integer into sums of smaller integers

4=1+1+1+1

4=2+1+1

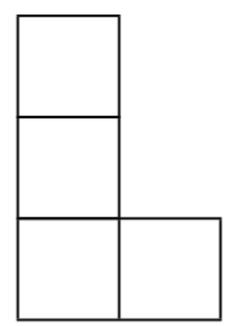


4=2+2

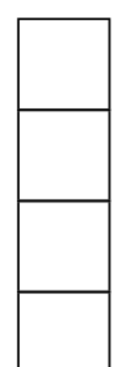


Young diagrams

4=3+1



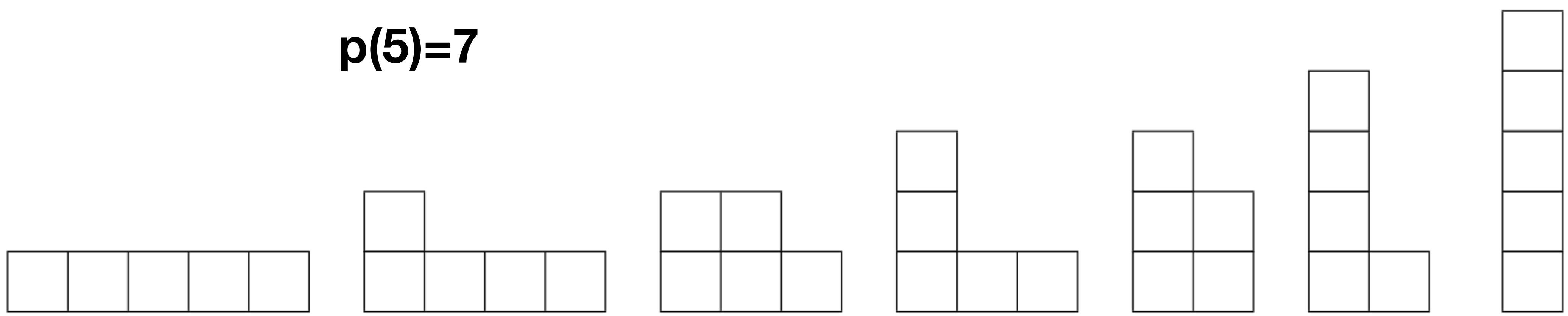
4=4+0



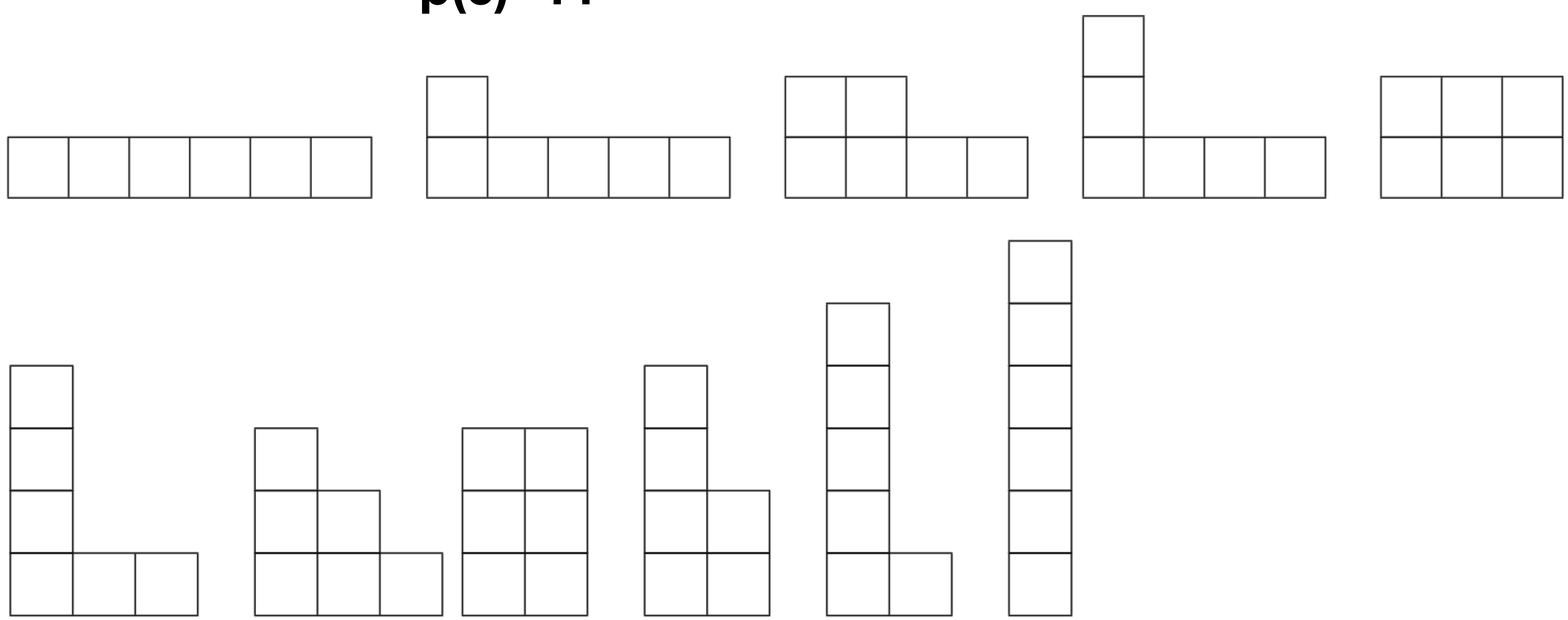
Partitions

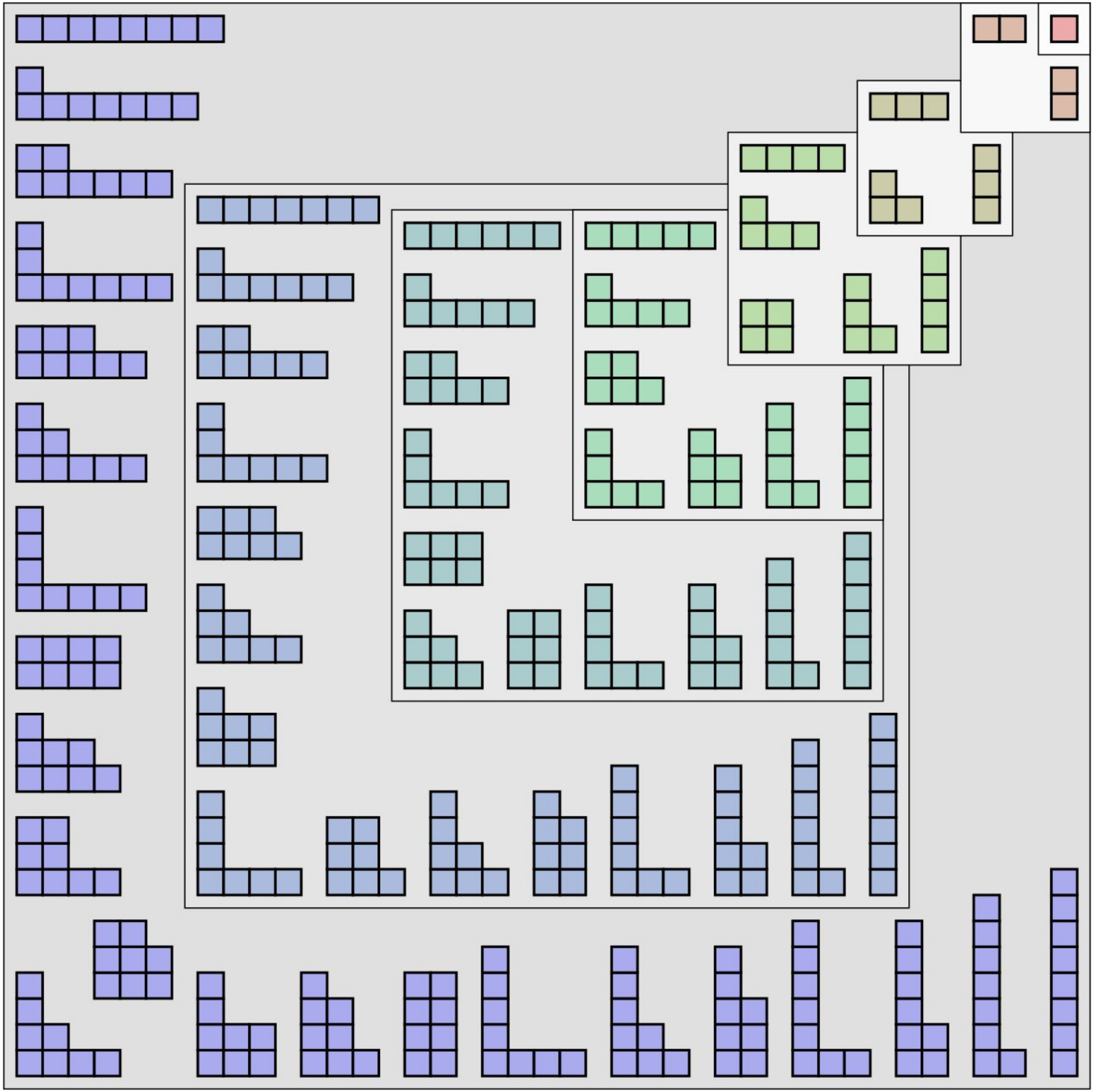
Problem: Find all partitions of numbers 1,2,3,4,5,6,7,8,9 together with their Young diagrams

n=5 **p(5)=7**



n=6 **p(6)=11**





Partitions

n=7 **p(7)=15**

{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {4, 1, 1, 1}, {3, 3, 1}, {3, 2, 2}, {3, 2, 1, 1},
{3, 1, 1, 1, 1}, {2, 2, 2, 1}, {2, 2, 1, 1, 1}, {2, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}

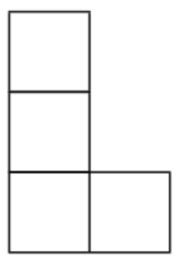
n=8 **p(8)=22**

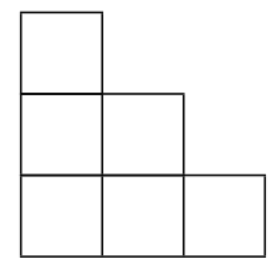
{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {5, 1, 1, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {4, 2, 1, 1},
{4, 1, 1, 1, 1}, {3, 3, 2}, {3, 3, 1, 1}, {3, 2, 2, 1}, {3, 2, 1, 1, 1}, {3, 1, 1, 1, 1, 1}, {2, 2, 2, 2},
{2, 2, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}

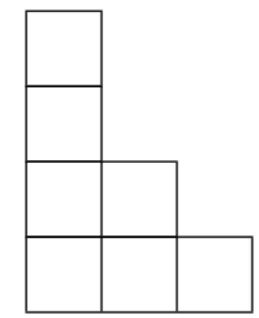
n=9 **p(9)=30**

{9}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1},
{5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1},
{3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1},
{3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1},
{1, 1, 1, 1, 1, 1, 1, 1, 1}

Odd & Distinct Parts







Problem (a): Count the number of partitions with ***odd parts*** from the previous examples

Problem (b): Count the number of partitions with ***distinct parts*** from the previous examples

n=7 **p(7)=15**

{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {4, 1, 1, 1}, {3, 3, 1}, {3, 2, 2}, {3, 2, 1, 1}, {3, 1, 1, 1, 1}, {2, 2, 2, 1}, {2, 2, 1, 1, 1}, {2, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}

n=8 **p(8)=22**

{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {5, 1, 1, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {4, 2, 1, 1}, {4, 1, 1, 1, 1}, {3, 3, 2}, {3, 3, 1, 1}, {3, 2, 2, 1}, {3, 2, 1, 1, 1}, {3, 1, 1, 1, 1, 1}, {2, 2, 2, 2}, {2, 2, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}

n=9 **p(9)=30**

{9}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}

Odd & Distinct Partitions

Find **odd and distinct** partitions for n = 1,2,...,11

n=9, p(9)=30

$\{9\}, \{8,1\}, \{7,2\}, \{7,1,1\}, \{6,3\}, \{6,2,1\}, \{6,1,1,1\}, \{5,4\}, \{5,3,1\}, \{5,2,2\}, \{5,2,1,1\},$
$\{5,1,1,1,1\}, \{4,4,1\}, \{4,3,2\}, \{4,3,1,1\}, \{4,2,2,1\}, \{4,2,1,1,1\}, \{4,1,1,1,1,1\}, \{3,3,3\},$
$\{3,3,2,1\}, \{3,3,1,1,1\}, \{3,2,2,2\}, \{3,2,2,1,1\}, \{3,2,1,1,1,1\}, \{3,1,1,1,1,1,1\}, \{2,2,2,2,1\}$
$\{2,2,2,1,1,1\}, \{2,2,1,1,1,1,1\}, \{2,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1\}$

n=10, p(10)=42

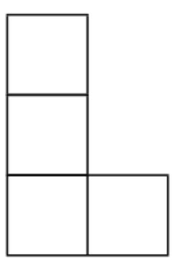
$\{\{10\}, \{9,1\}, \{8,2\}, \{8,1,1\}, \{7,3\}, \{7,2,1\}, \{7,1,1,1\}, \{6,4\}, \{6,3,1\}, \{6,2,2\}, \{6,2,1,1\},$
$\{6,1,1,1,1\}, \{5,5\}, \{5,4,1\}, \{5,3,2\}, \{5,3,1,1\}, \{5,2,2,1\}, \{5,2,1,1,1\}, \{5,1,1,1,1,1\},$
$\{4,4,2\}, \{4,4,1,1\}, \{4,3,3\}, \{4,3,2,1\}, \{4,3,1,1,1\}, \{4,2,2,2\}, \{4,2,2,1,1\}, \{4,2,1,1,1,1\},$
$\{4,1,1,1,1,1,1\}, \{3,3,3,1\}, \{3,3,2,2\}, \{3,3,2,1,1\}, \{3,3,1,1,1,1\}, \{3,2,2,2,1\}, \{3,2,2,1,1,1\}$
$\{3,2,1,1,1,1,1\}, \{3,1,1,1,1,1,1,1\}, \{2,2,2,2,2\}, \{2,2,2,2,1,1\}, \{2,2,2,1,1,1,1\},$
$\{2,2,1,1,1,1,1,1\}, \{2,1,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1,1\}\}$

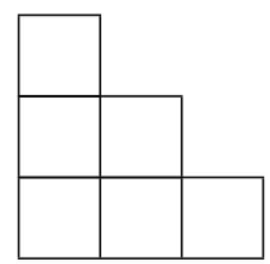
n=11, p(11)=56

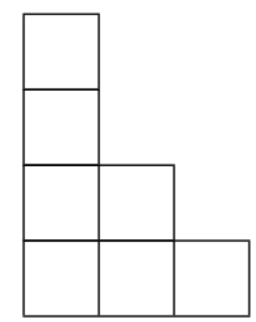
$$\{\{11\},\{10,1\},\{9,2\},\{9,1,1\},\{8,3\},\{8,2,1\},\{8,1,1,1\},\{7,4\},\{7,3,1\},\{7,2,2\},\{7,2,1,1\},$$
$$\{7,1,1,1,1\},\{6,5\},\{6,4,1\},\{6,3,2\},\{6,3,1,1\},\{6,2,2,1\},\{6,2,1,1,1\},\{6,1,1,1,1,1\},\{5,5,1\},$$
$$\{5,4,2\},\{5,4,1,1\},\{5,3,3\},\{5,3,2,1\},\{5,3,1,1,1\},\{5,2,2,2\},\{5,2,2,1,1\},\{5,2,1,1,1,1\},$$
$$\{5,1,1,1,1,1,1\},\{4,4,3\},\{4,4,2,1\},\{4,4,1,1,1\},\{4,3,3,1\},\{4,3,2,2\},\{4,3,2,1,1\},\{4,3,1,1,1,1\},$$
$$\{4,2,2,2,1\},\{4,2,2,1,1,1\},\{4,2,1,1,1,1,1\},\{4,1,1,1,1,1,1,1\},\{3,3,3,2\},\{3,3,3,1,1\},\{3,3,2,2,1\}$$
$$\{3,3,2,1,1,1\},\{3,3,1,1,1,1,1\},\{3,2,2,2,2\},\{3,2,2,2,1,1\},\{3,2,2,1,1,1,1\},\{3,2,1,1,1,1,1,1\},$$
$$\{3,1,1,1,1,1,1,1,1\},\{2,2,2,2,2,1\},\{2,2,2,2,1,1,1\},\{2,2,2,1,1,1,1,1\},\{2,2,1,1,1,1,1,1,1\},$$
$$\{2,1,1,1,1,1,1,1,1,1\},\{1,1,1,1,1,1,1,1,1,1,1\}\}$$

Odd vs. Distinct

n	1	2	3	4	5	6	7	8	9
p(n)	1	2	3	5	7	11	15	22	30
# odd	1	1	2	2	3	4	5	6	8
# dist.	1	1	2	2	3	4	5	6	8







Problem: Why is the number of these partitions is the same for every n?

odd	distinct
5	5
3,1,1	4,1
1,1,1,1,1	3,2

odd	distinct
5,1	6
3,3	5,1
3,1,1,1	4,2
1,1,1,1,1,1	3,2,1

odd	distinct
7	7
5,1,1	6,1
3,3,1	5,2
3,1,1,1,1	4,3
1,1,1,1,1,1,1	4,2,1

odd	distinct
7,1	8
5,3	7,1
5,1,1,1	6,2
3,3,1,1	5,3
3,1,1,1,1,1	5,2,1
1,1,1,1,1,1,1,1	4,3,1

odd	distinct
9	9
7,1,1	8,1
5,1,1,1	7,2
5,3,1	6,3
3,3,3	6,2,1
3,3,1,1,1	5,4
3,1,1,1,1,1,1	5,3,1
1,1,1,1,1,1,1,1,1	4,3,2

Matching

From Distinct to Odd.

$$2 \to 1, 1$$
$$4 \to 1, 1, 1, 1$$
$$6 \to 3, 3$$
$$8 \to 1, 1, 1, 1, 1, 1, 1, 1$$

$$8 \to 1, 1, 1, 1, 1, 1, 1, 1$$
$$7, 1 \to 7, 1$$
$$6, 2 \to 3, 3, 1, 1$$
$$5, 3 \to 5, 3$$
$$5, 2, 1 \to 5, 1, 1, 1$$
$$4, 3, 1 \to 3, 1, 1, 1, 1, 1$$

From Odd to Distinct.

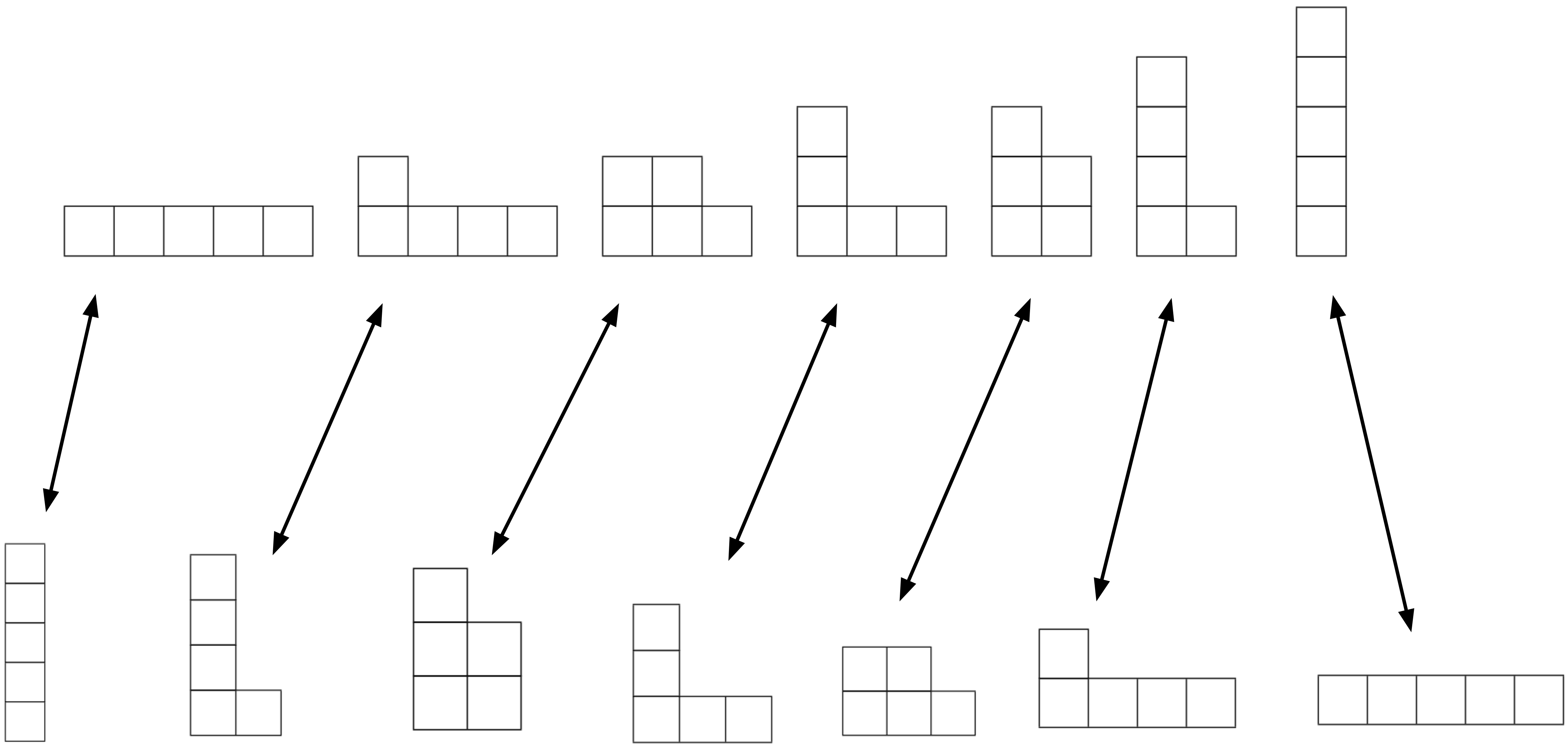
$$1 \to 1$$
$$1, 1 \to 2$$
$$1, 1, 1 \to 2, 1$$
$$1, 1, 1, 1 \to 4$$
$$1, 1, 1, 1, 1 \to 4, 1$$
$$1, 1, 1, 1, 1, 1 \to 4, 2$$
$$1, 1, 1, 1, 1, 1, 1 \to 4, 2, 1$$
$$1, 1, 1, 1, 1, 1, 1, 1 \to 8$$
$$1, 1, 1, 1, 1, 1, 1, 1, 1 \to 8, 1$$

Binary presentation

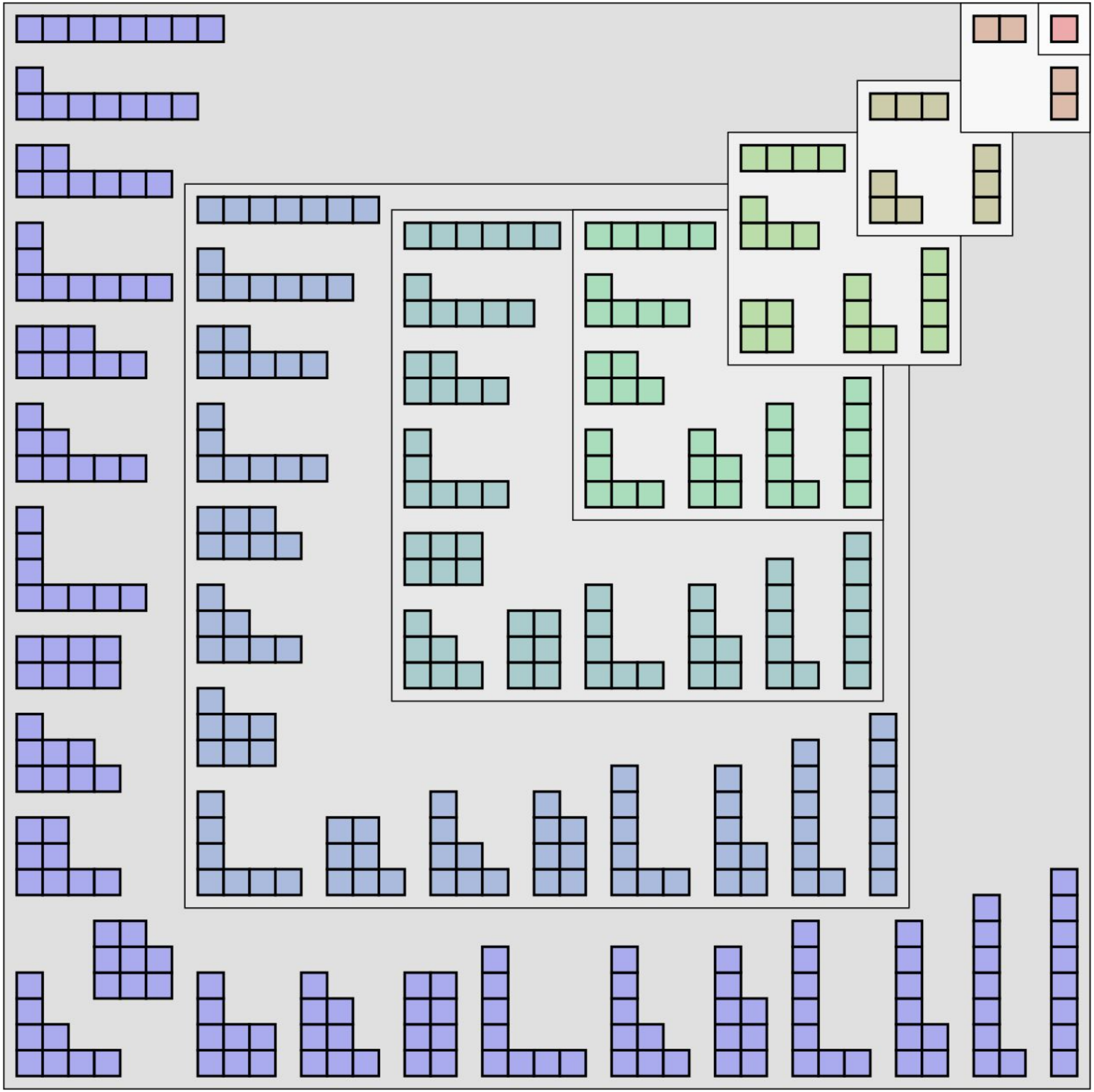
$$3 = 2^1 + 2^0, \qquad 5 = 2^2 + 2^0, \qquad 8 = 2^3$$

Conjugated Partitions

Flip Young diagrams over the diagonal



Problem: how many self-conjugated (symmetric) partitions are there?



p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

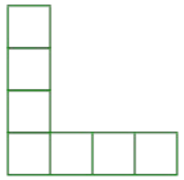
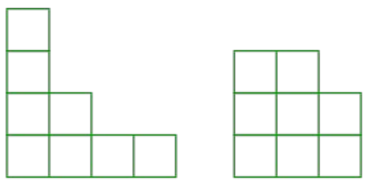
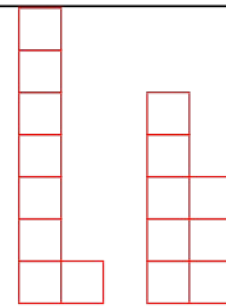
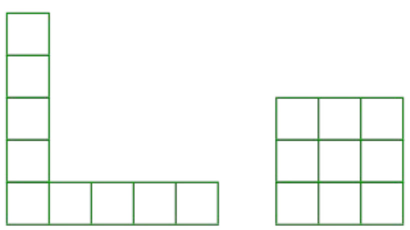
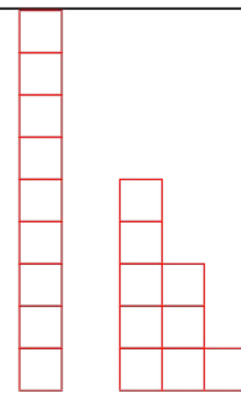
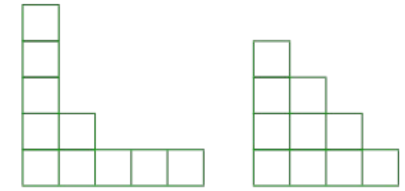
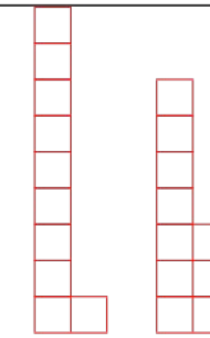
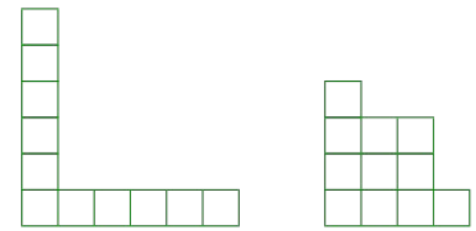
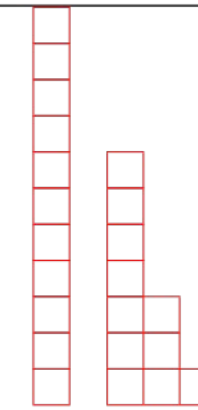
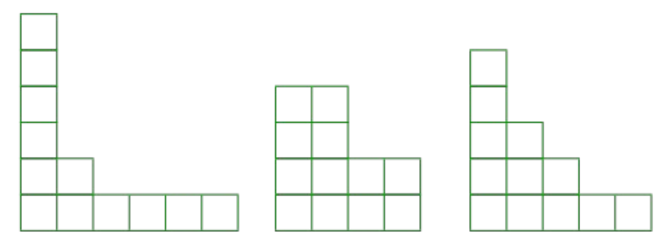
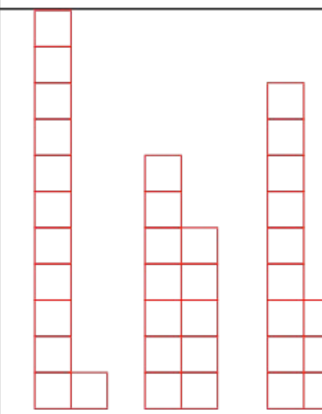
p(11)

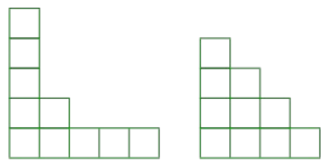
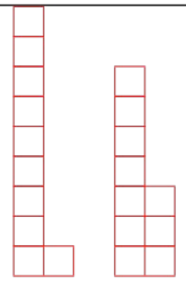
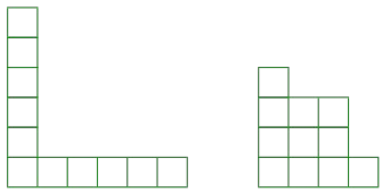
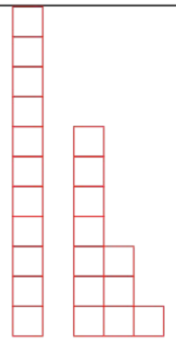
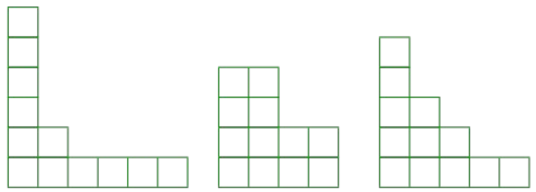
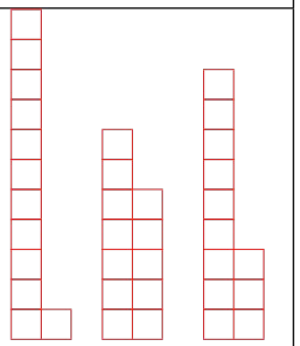
{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

Odd, Distinct, Symmetric

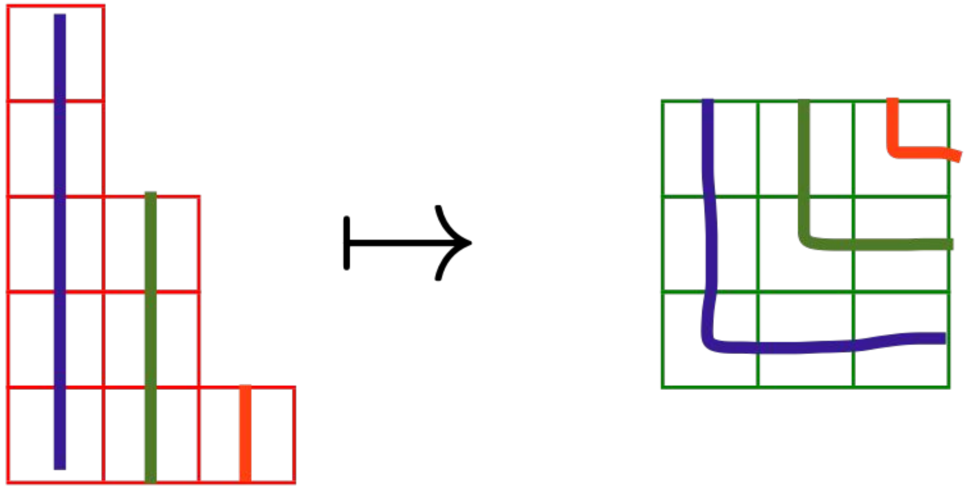
n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Problem: Show that Symmetric = Odd&Distinct

n	Symmetric	Odd&Distinct
7		
8		
9		
10		
11		
12		

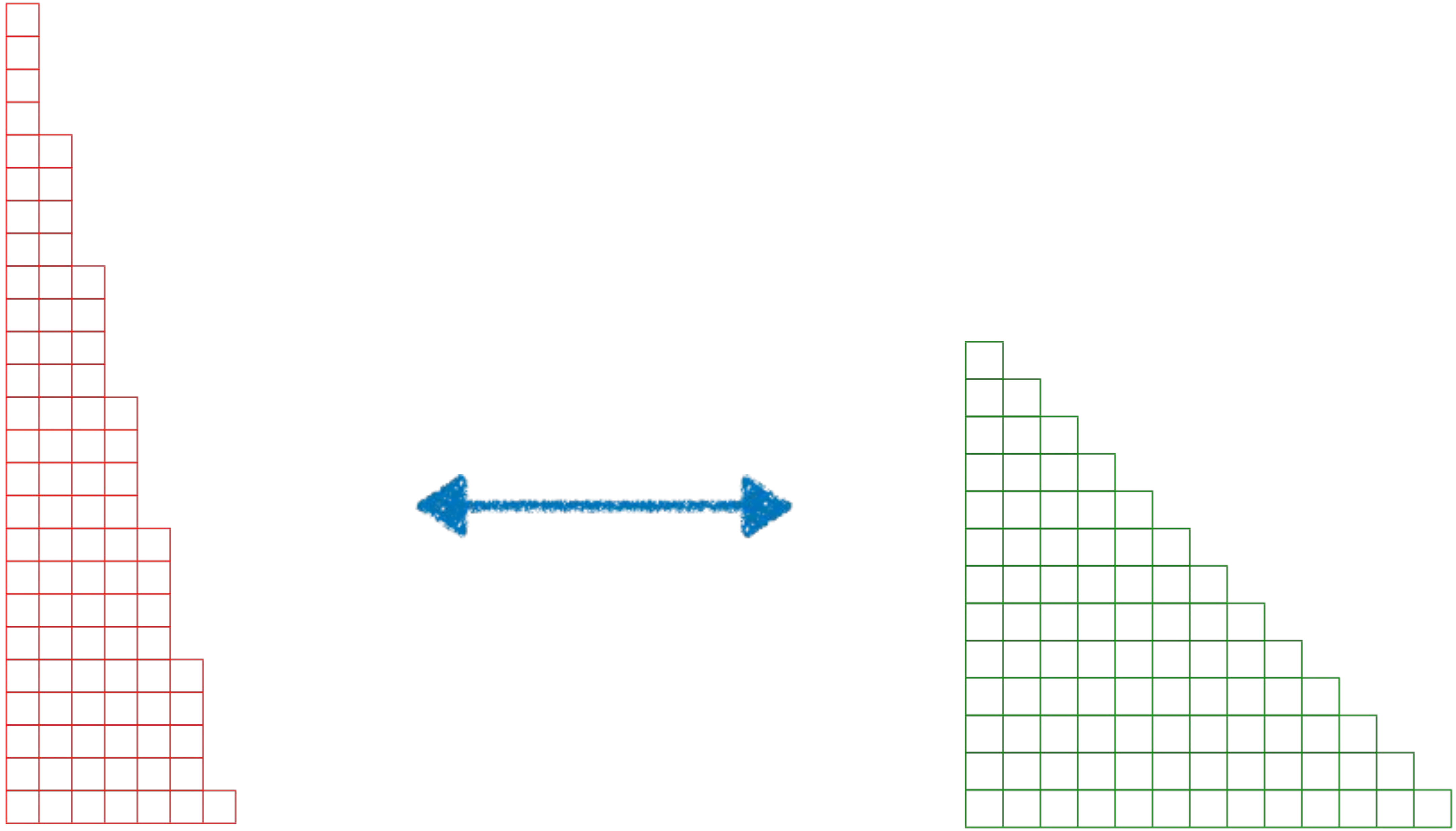
10		
11		
12		

Matching



Odd&Distinct vs Symmetric

Triangular numbers

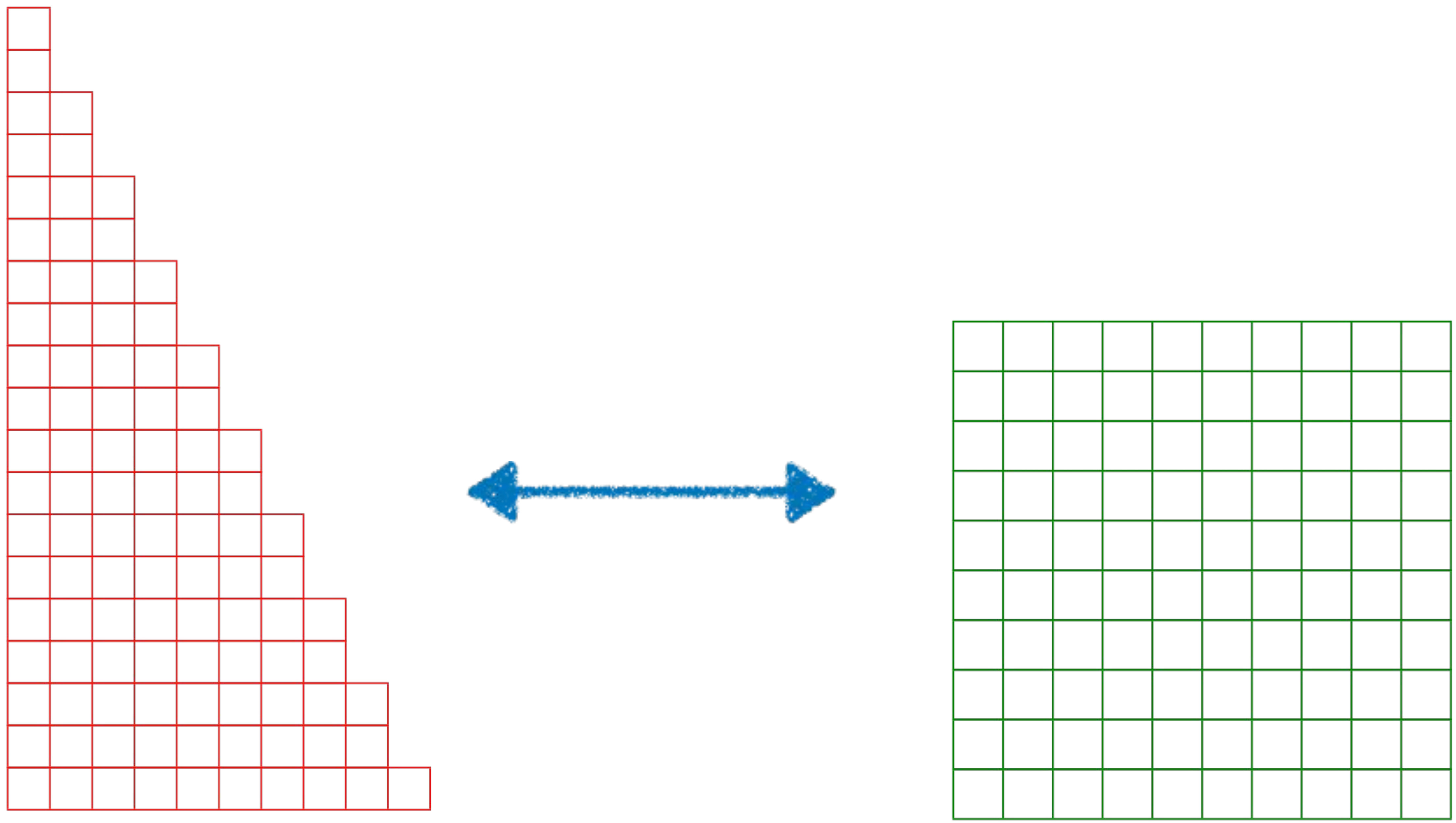


{25,21,17,13,9,5,1}

{13,12,11,10,9,8,7,6,5,4,3,2,1

Odd&Distinct vs Symmetric

Square numbers



$\{19, 17, 15, 13, 11, 9, 7, 5, 3, 1\}$

$\{10, 10, 10, 10, 10, 10, 10, 10, 10, 10\}$

Computing Sums

Compute the sum. How many terms are in the sum?

- $1 + 3 + 5 + \ldots + 117 + 119 = ?$

- $1 + 5 + 9 + 13 + 17 + 21 + 25 = ?$

- $1 + 5 + 9 + 13 + \ldots + 81 = ?$

Divisibility by 3

Consider partitions of n whose parts are not divisible by **3**

Compare those with partitions of n in which **each part** is not repeated **3 or more times**

Divisibility by 4

Consider partitions of n whose parts are not divisible by **4**

Compare those with partitions of n in which **each part** is not repeated **4 or more times**

Divisibility by n

Consider partitions of n whose parts are not divisible by **n**

Compare those with partitions of n in which **each part** is not repeated **n or more times**

Restricted Partitions

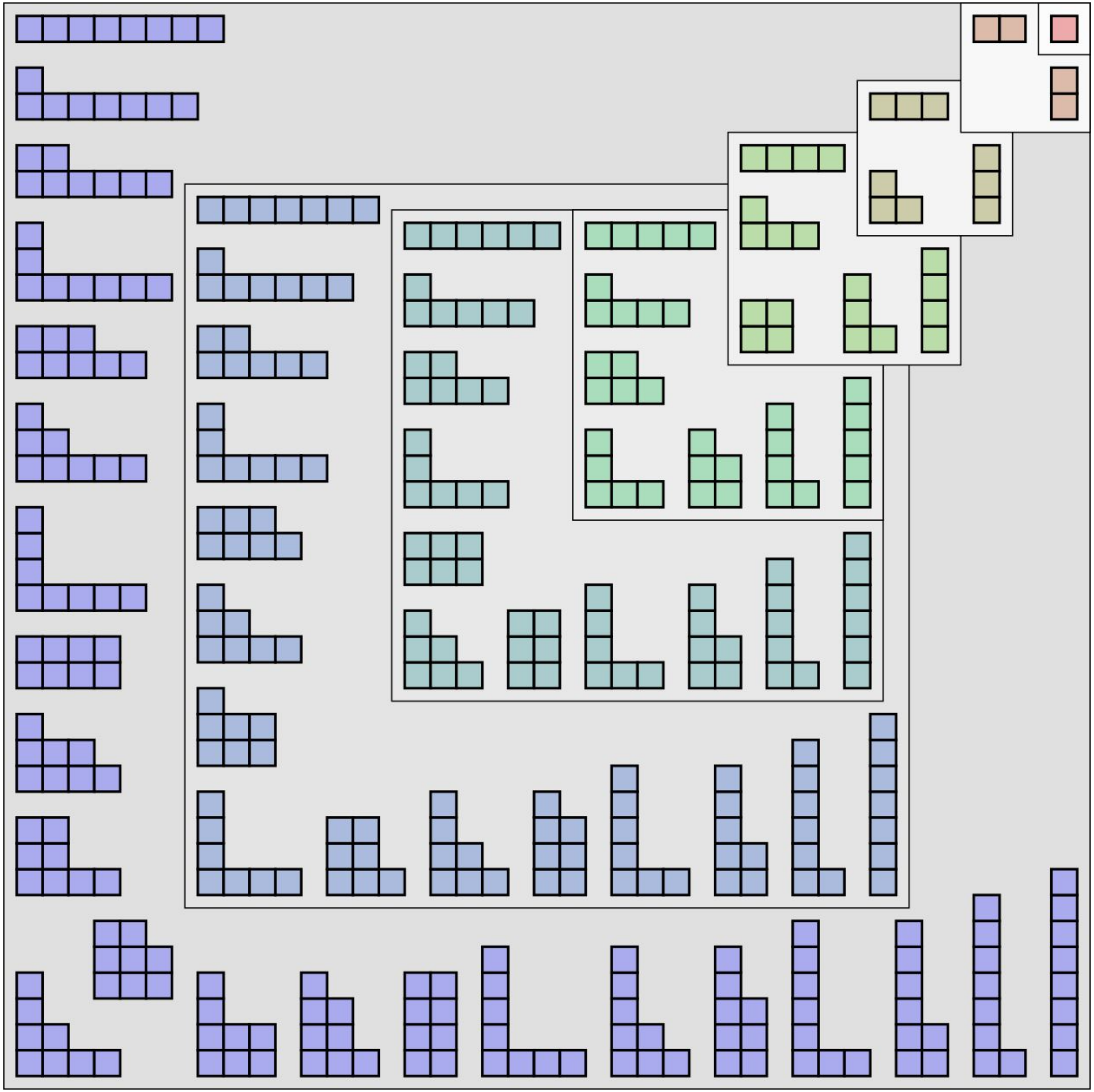
Restricted Partitions. Let us now look at integer partitions of n which have *exactly* 4 parts. From the list of partitions of 7

$$\{\{7\},\{6,1\},\{5,2\},\{5,1,1\},\{4,3\},\{4,2,1\},\{4,1,1,1\},\{3,3,1\},\{3,2,2\},$$
$$\{3,2,1,1\},\{3,1,1,1,1\},\{2,2,2,1\},\{2,2,1,1,1\},\{2,1,1,1,1,1\},\{1,1,1,1,1,1,1\}\}$$

Only these qualify

$$\{\{4,1,1,1\},\{3,2,1,1\},\{2,2,2,1\}\}$$

Make lists for $n = 4, 5, 6$ with all possible restricted parts, i.e. all partitions of 4 with 1, with 2, with 3 parts, etc., same for $n = 5$ and $n = 6$. Count each number of partitions, call it $p_k(n)$. Do you see any pattern?



	1	2	3	4	5	6	7	8
p(1)	1							
p(2)	1	1						
p(3)	1	1	1					
p(4)	1	2	1	1				
p(5)	1	2	2	1	1			
p(6)	1	3	3	2	1	1		
p(7)	1	3	4	3	2	1	1	
p(8)	1	4	5	5	3	2	1	1

p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(11)

{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

	1	2	3	4	5	6	7	8	9	10	11	12
p(1)	1											
p(2)	1	1										
p(3)	1	1	1									
p(4)	1	2	1	1								
p(5)	1	2	2	1	1							
p(6)	1	3	3	2	1	1						
p(7)	1	3	4	3	2	1	1					
p(8)	1	4	5	5	3	2	1	1				
p(9)	1	4	7	6	5	3	2	1	1			
p(10)	1	5	8	9	7	5	3	2	1	1		
p(11)												
p(12)												

Partitions of n

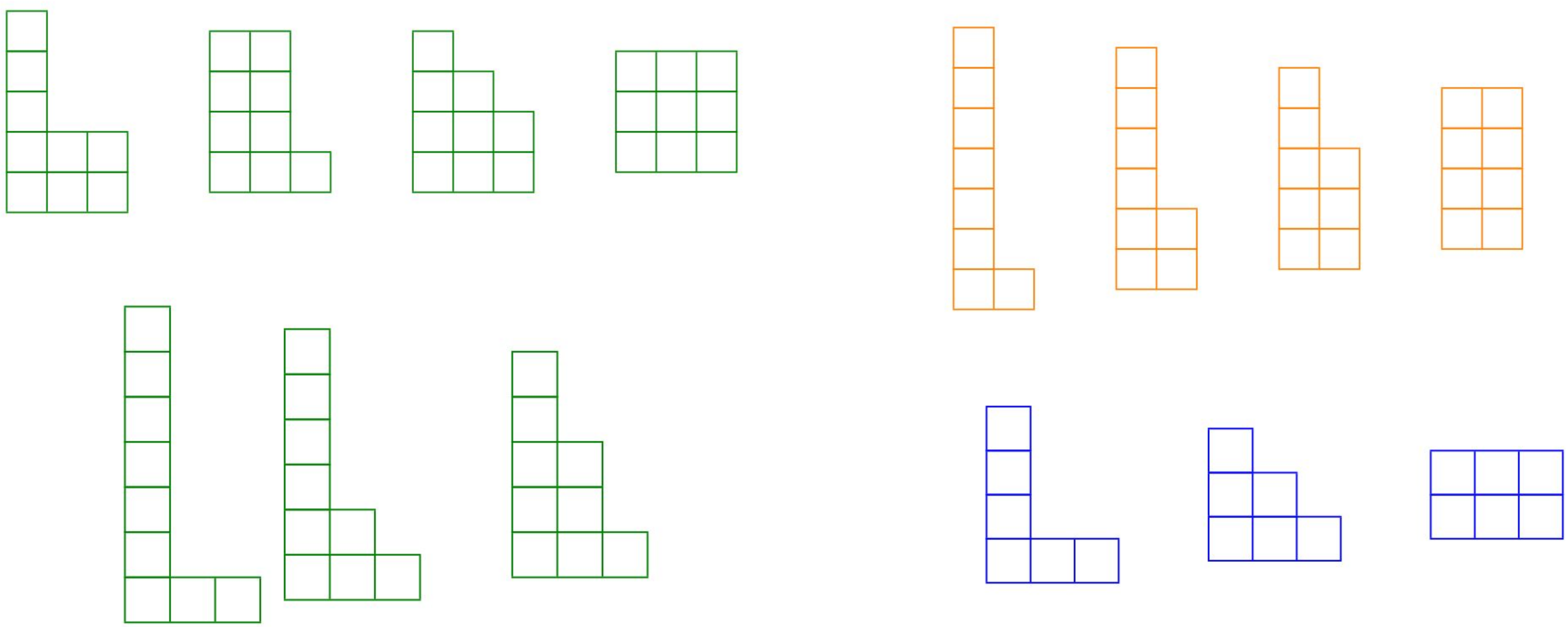
n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Recurrent Formula

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$

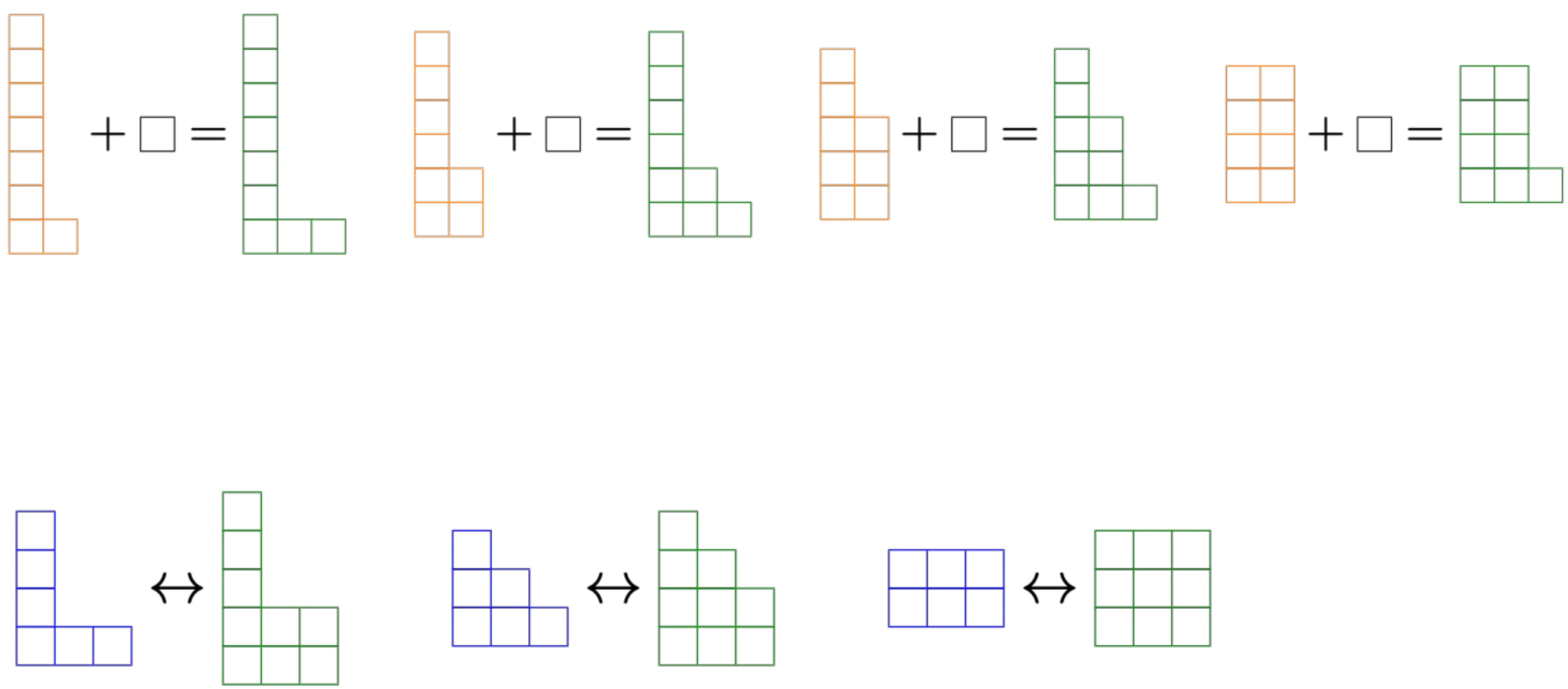
$$p_3(9) = p_2(8) + p_3(6)$$

7 4 3



Matching

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$



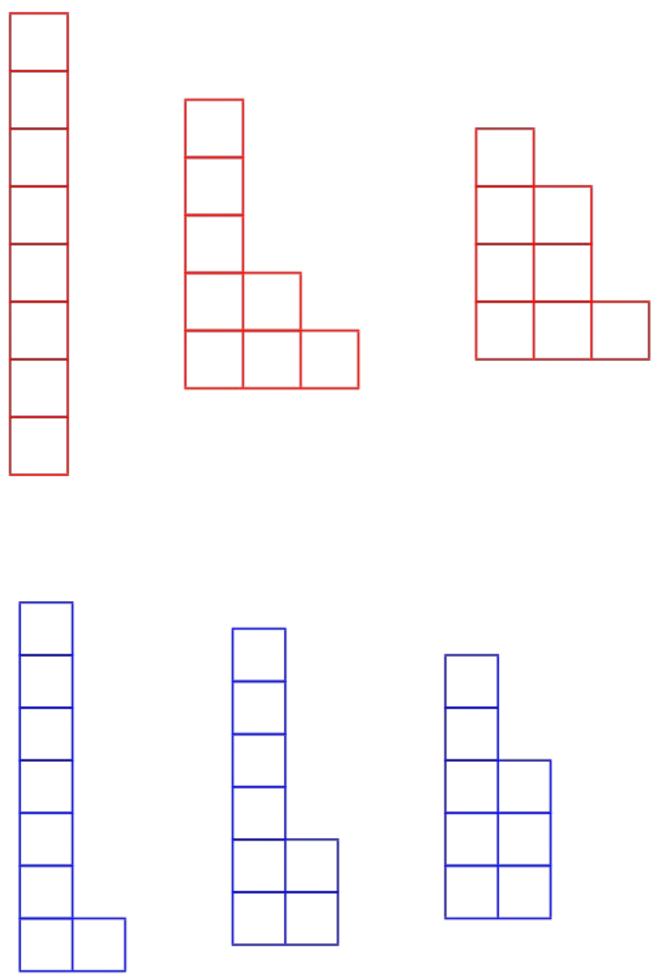
Understanding p(n)

n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

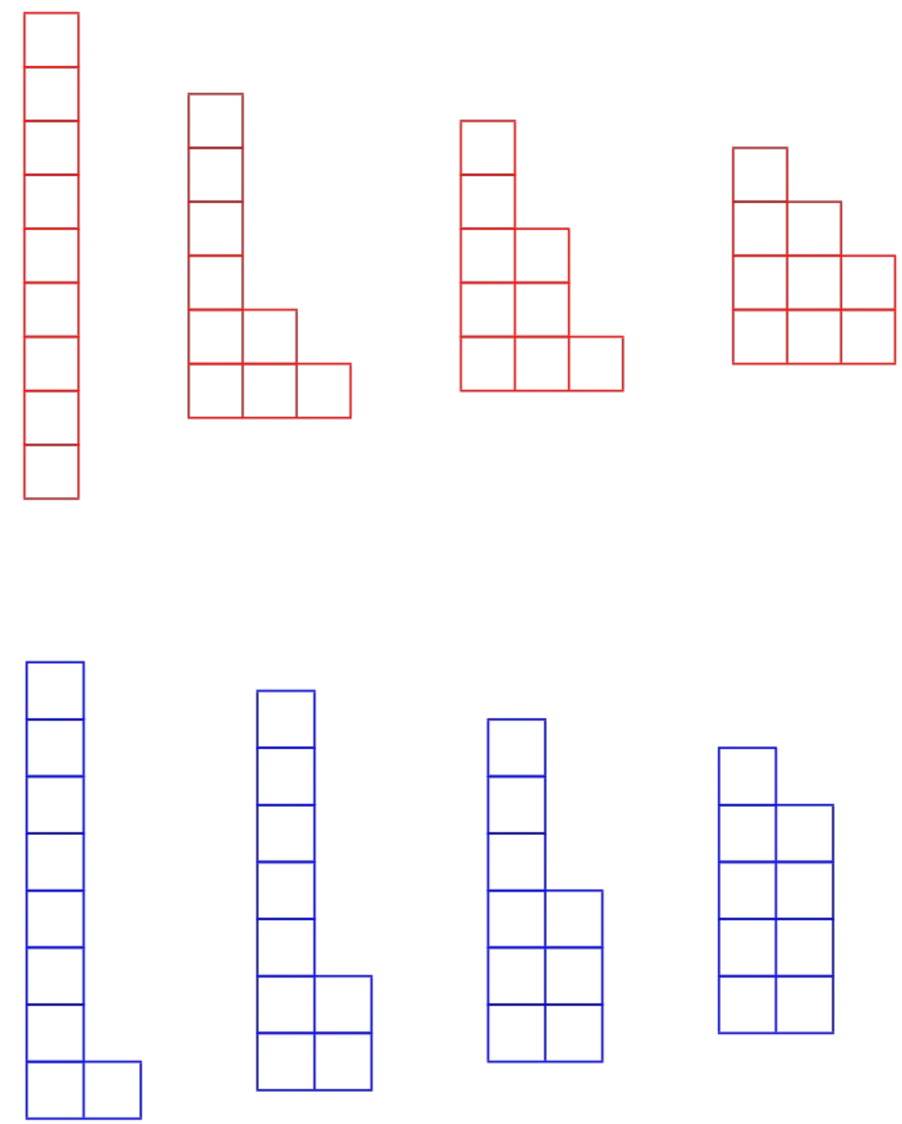
Odd and Even Number of parts

For each **n** count *distinct partitions* with **odd** and **even** *number of parts*

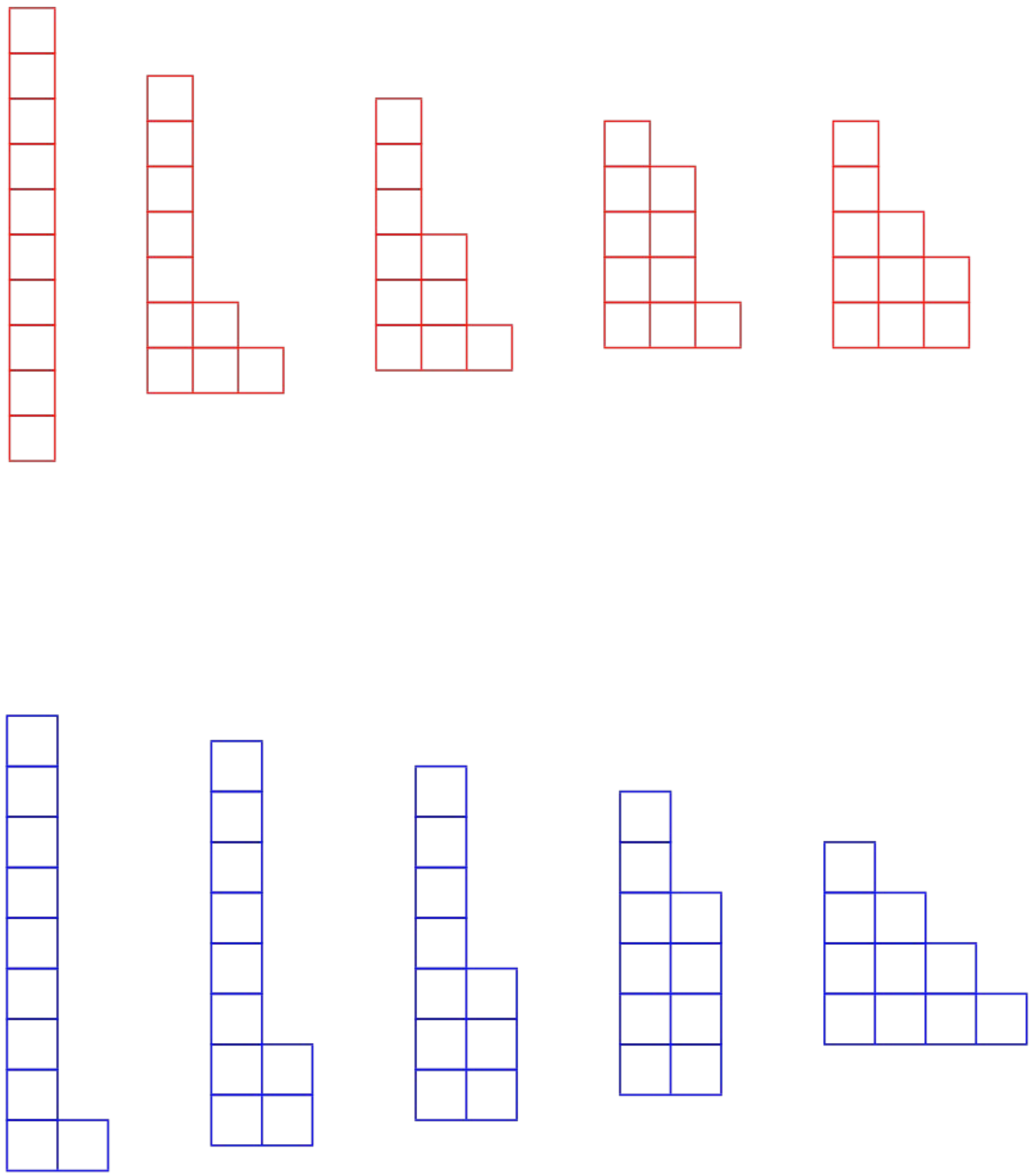
n=8



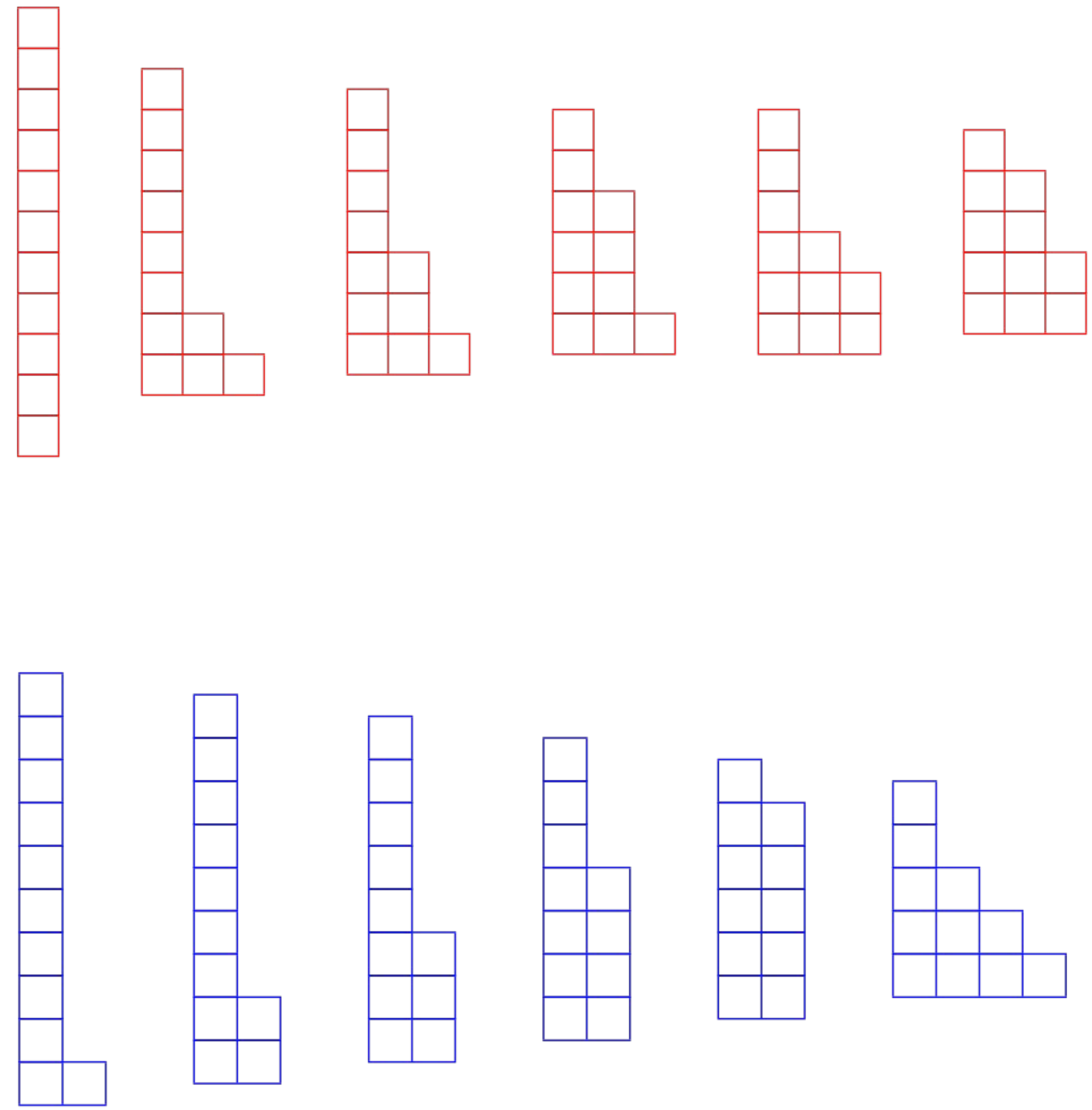
n=9



n=10

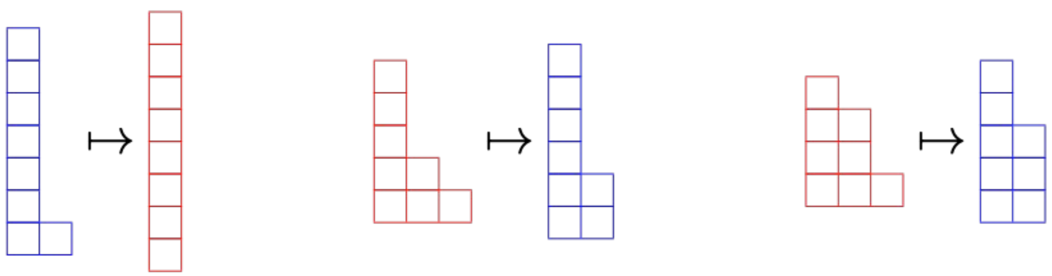


n=11

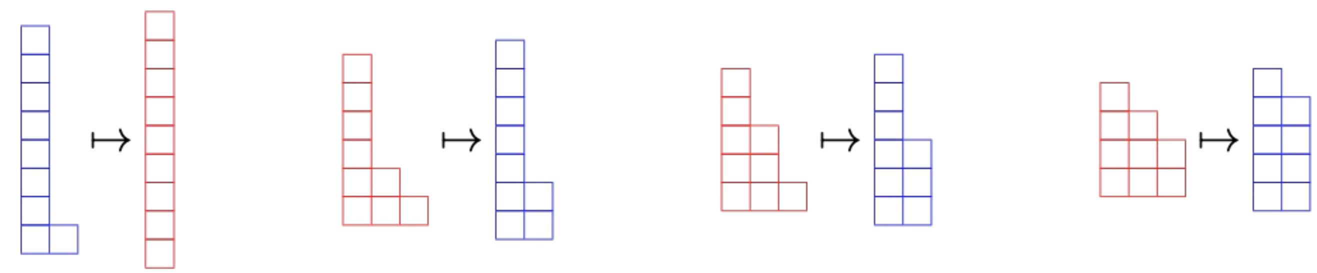


Matching

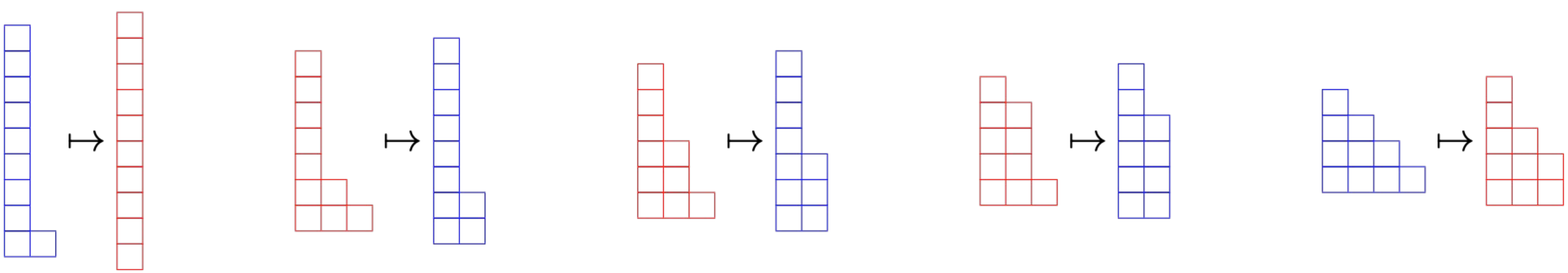
n=8



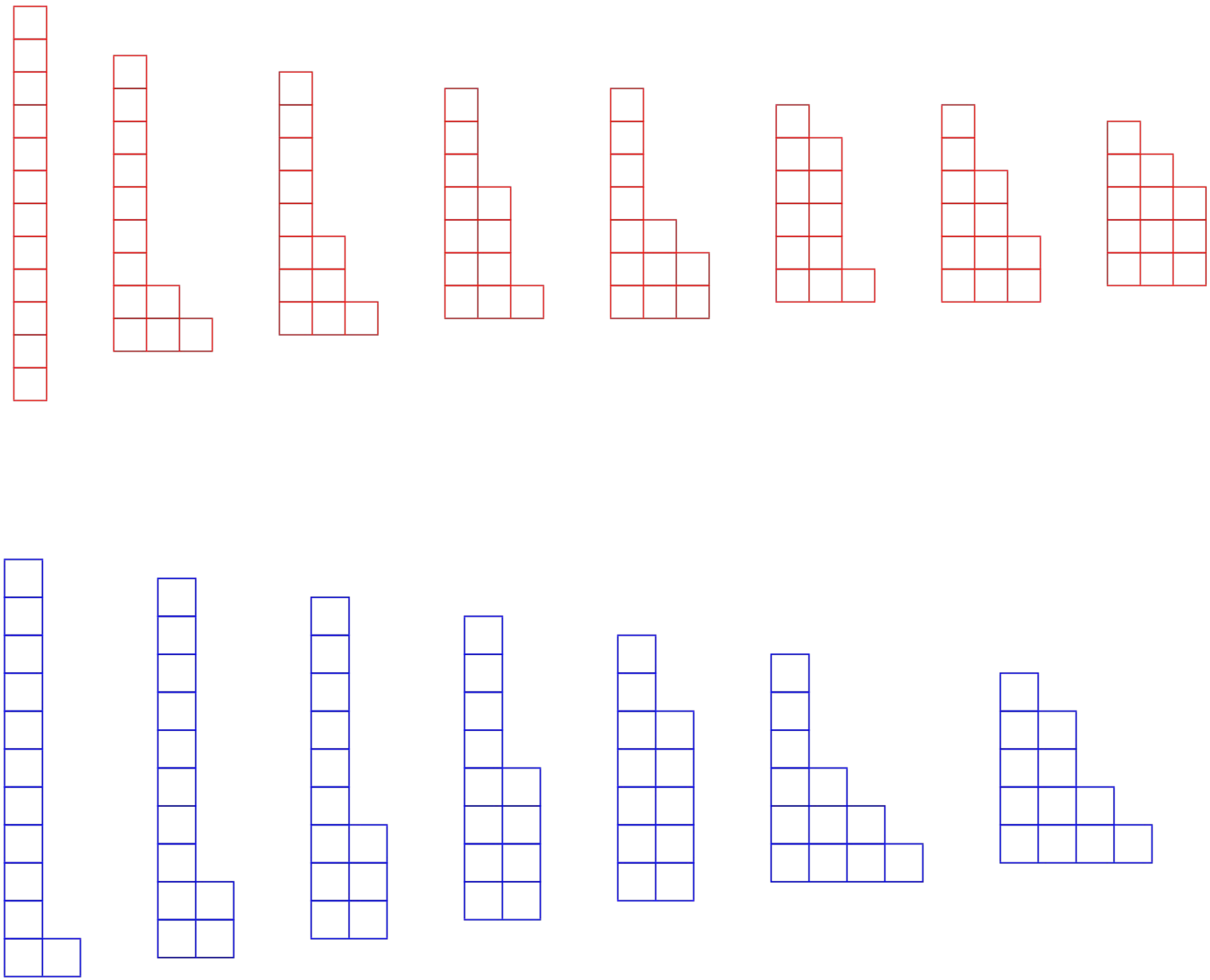
n=9



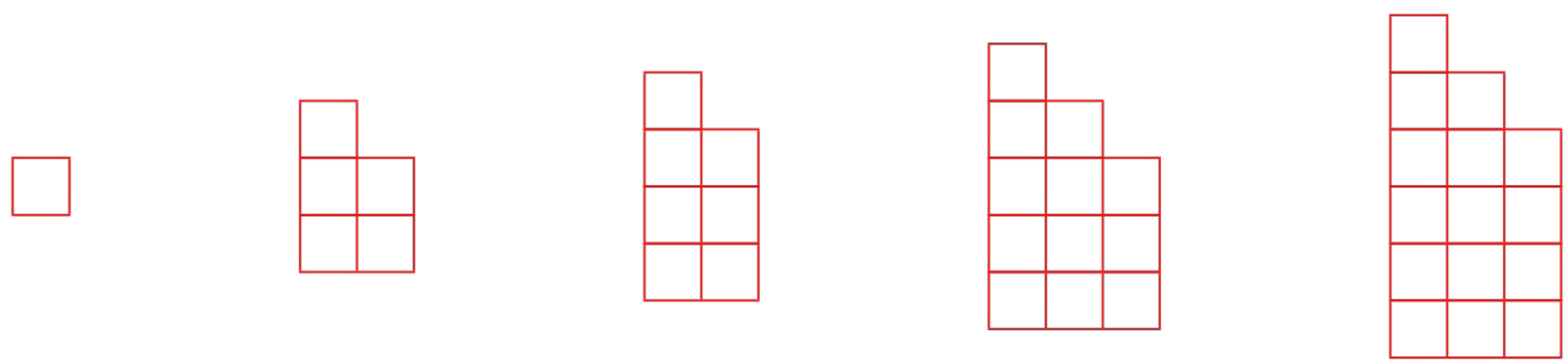
n=10

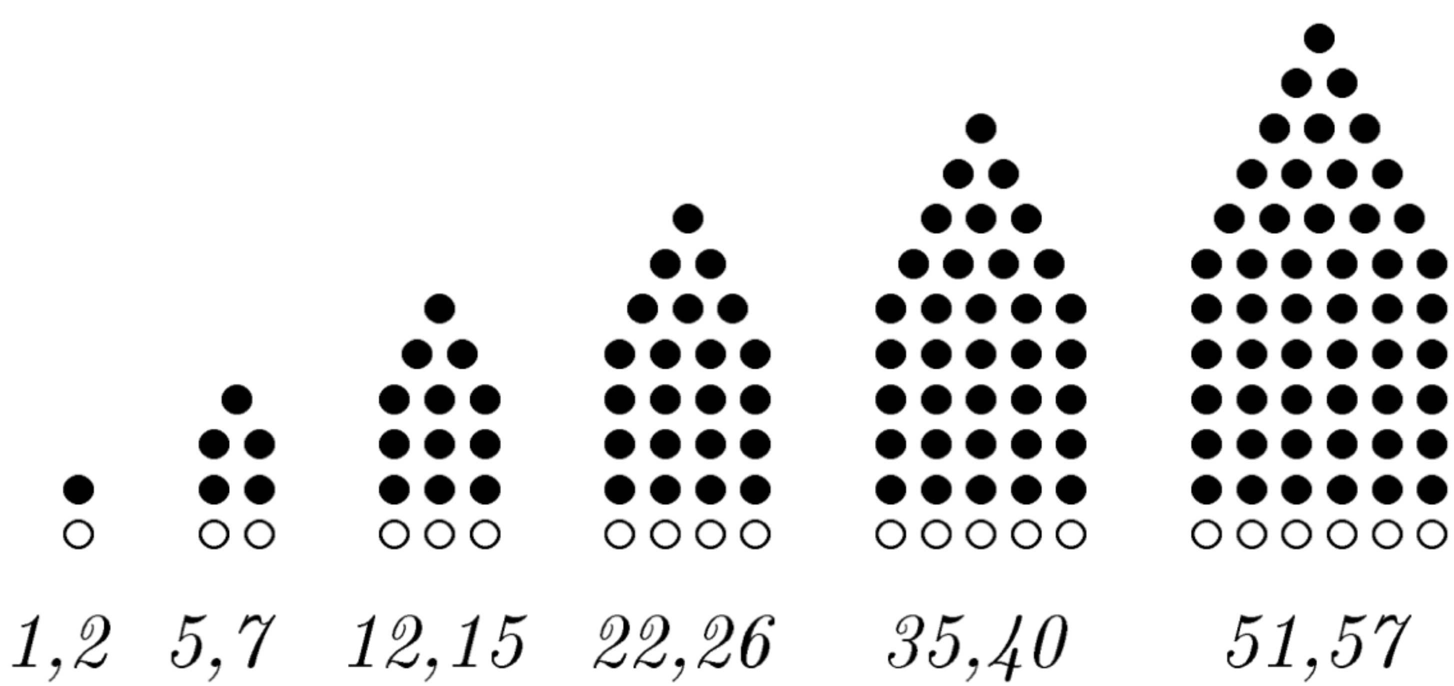


n=12



Pentagonal Numbers



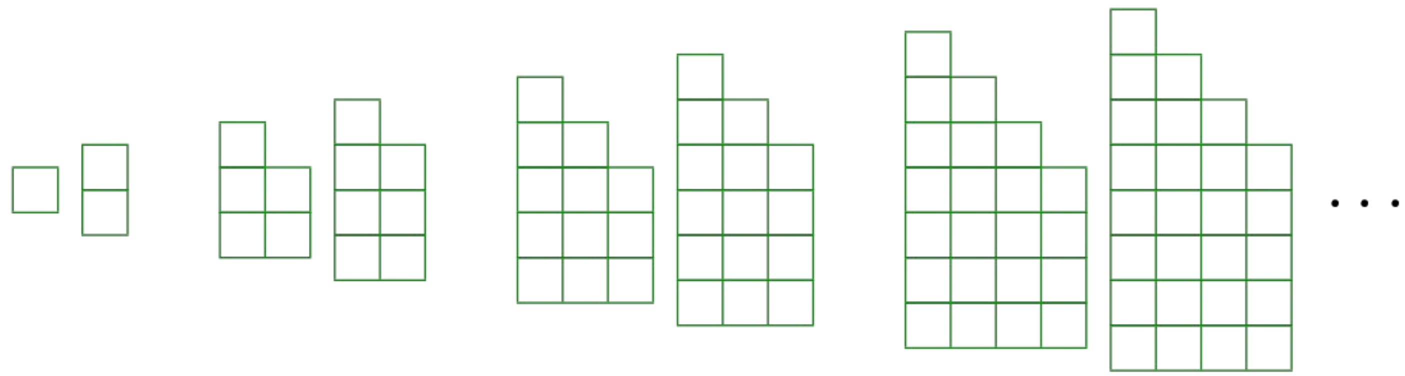


Pentagonal Numbers

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
T_n	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
S_n	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64
P_n	100	77	57	40	26	15	7	2	0	1	5	12	22	35	51	70	92

1,2, 5,7, 12,15, 22,26, 35,40, 51,57, 70,77,
92,100,

Pentagonal Numbers

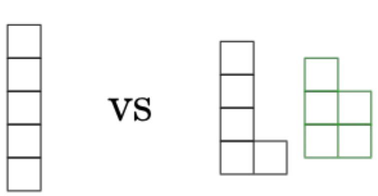


Show that $P_n = \frac{n(3n-1)}{2}$

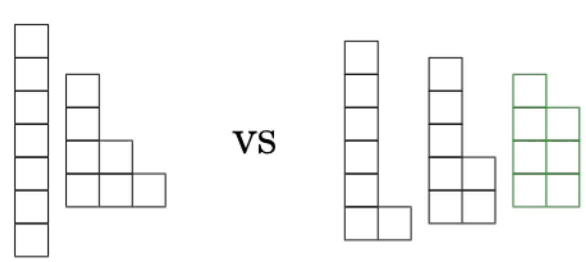
$(P_1, P_{-1}) \quad (P_2, P_{-2}) \quad (P_3, P_{-3}) \quad (P_4, P_{-4}) \quad \ldots$

Theorem: If n is not a pentagonal number, then the number of even distinct partitions of n, call it $q_e(n)$ equals the number of odd distinct partitions of n, call it $q_o(n)$. So $q_e(n) = q_o(n)$ and so the total number of distinct partitions of n, call it $q(n)$ is $q(n) = 2q_o(n)$ which is even.

If n is a pentagonal number, say $n = P_j$, then $q_e(n) = q_o(n) + (-1)^j$ and so $q(n) = 2q_o(n) + (-1)^j$ which is odd.



n=5



n=7

Generating Function for Partitions

$$(1-z)(1-z^2) = 1\times(1-z^2)-z\times(1-z^2) = 1\times1-1\times z^2-z\times1-z\times(-z^2) = 1-z-z^2+z^3$$

$$(1-z)(1-z^2)(1-z^3) = (1-z-z^2+z^3)(1-z^3) = (1-z-z^2+z^3-z^3)1+(1-z-z^2+z^3)(-z^3)$$
$$= 1-z-z^2+z^3-z^3-z^3+z^4+z^5-z^6 = 1-z-z^2+z^4+z^5-z^6$$

$$\phi(z) = \prod_{k=1}^{\infty}(1-z^k) = (1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots$$

$$\phi(z) = 1-z^1-z^2+z^5+z^7-z^{12}-z^{15}+z^{22}+z^{26}-\ldots$$

coefficient for z^n equals: #distinct even partitions of n
-#distinct odd partitions of n

$$\mathbf{p}(z) = \frac{1}{\phi(z)} = \prod_{k=1}^{\infty}\frac{1}{1-z^k} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots}$$

Recall that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \dots$$

so

$$\frac{1}{1-z^k} = 1 + z^k + z^{2k} + z^{3k} + z^{4k} + \dots$$

Generating function

$$\mathbf{p}(z) = \frac{1}{\phi(z)} = \prod_{k=1}^{\infty} \frac{1}{1-z^k} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\dots}$$

$$\mathbf{p}(z) = (1 + z + z^2 + z^3 + \dots)(1 + z^2 + z^4 + z^6 + \dots)(1 + z^3 + z^6 + z^9 + \dots)\cdot\dots$$

$$\mathbf{p}(z) = \prod_{k=1}^{\infty}(1 + z^k + z^{2k} + z^{3k} + \dots)$$

$$\mathbf{p}(z) = (1 + z + z^2 + z^3 + \dots)(1 + z^2 + z^4 + z^6 + \dots)(1 + z^3 + z^6 + z^9 + \dots) \cdot \dots$$

Now we need to collect terms in front of each power of z. Each term z^n in the resulting product will look like

$$z^{k_1} \cdot z^{2k_2} \cdot z^{3k_3} \cdot \dots \cdot z^{mk_m} = z^{k_1 + 2k_2 + 3k_3 + \dots + mk_m}$$

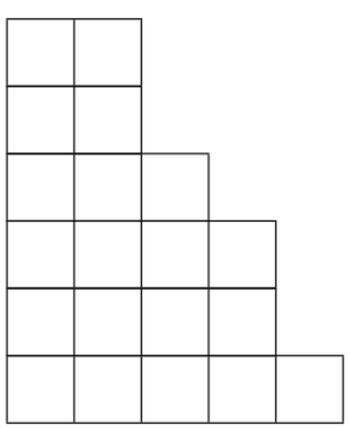
We want to count the number of such products with $k_1 + 2k_2 + 3k_3 + \dots + mk_m = n$

$$n = k_1 + 2k_2 + \dots + mk_m = \underbrace{1 + \dots + 1}_{k_1} + \underbrace{2 + \dots + 2}_{k_2} + \dots + \underbrace{m + \dots + m}_{k_m}$$

which is the number of partitions of n $\quad \{\underbrace{m, \dots, m}_{k_m}, \underbrace{m-1, \dots, m-1}_{k_{m-1}}, \dots, \underbrace{2, \dots, 2}_{k_2}, \underbrace{1, \dots, 1}_{k_1}\}$

consider partition $\{6, 6, 4, 3, 1\}$ of 20

$m = 6$ and $k_6 = 2, k_5 = 0, k_4 = 1, k_3 = 1, k_2 = 0, k_1 = 1.$



$$\mathbf{p}(z) = 1 + p(1)z + p(2)z^2 + p(3)z^3 + p(4)z^4 +$$

$$\mathbf{p}(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 11z^6 + 15z^7 + 22z^8 + 30z^9 + 42z^{10} + 56z^{11} + 77z^{12} + 101z^{13} + \dots$$

"Read Euler, read Euler, he is the master of us all."

P. Laplace

{1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627}

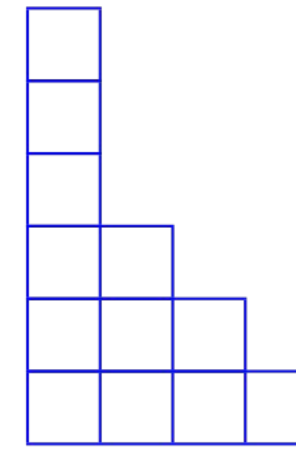
$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-p(n-22)-p(n-26)+\ldots$$

$$\begin{aligned} 5 &= 3 + 2\,, \\ 11 &= 7 + 5 - 1\,, \\ 15 &= 11 + 7 - 2 - 1\,, \\ 56 &= 42 + 30 - 11 - 5\,, \\ 77 &= 56 + 42 - 15 - 7 + 1\,, \\ 101 &= 77 + 56 - 22 - 11 + 1\,. \end{aligned}$$

Partition Function

$$(1-z)(1-z^2) = 1 - z - z^2 + z^3$$

$$\phi(z) = (1-z)(1-z^2)(1-z^3)(1-z^4)\cdots$$



terms will look like $(-1)^k z^{n_1+n_2+\cdots+n_k}$

coefficient for z^n equals: #distinct even partitions of n
-#distinct odd partitions of n

Partition function for p(n)

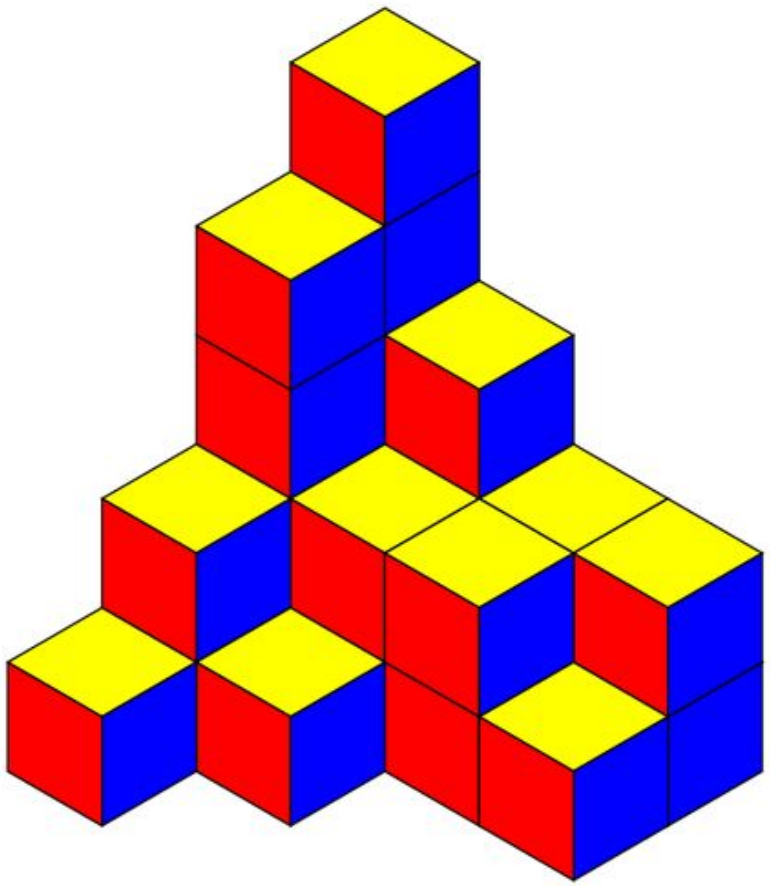
$$p(z) = \prod_{k=1}^{\infty} \frac{1}{(1-z^k)}$$

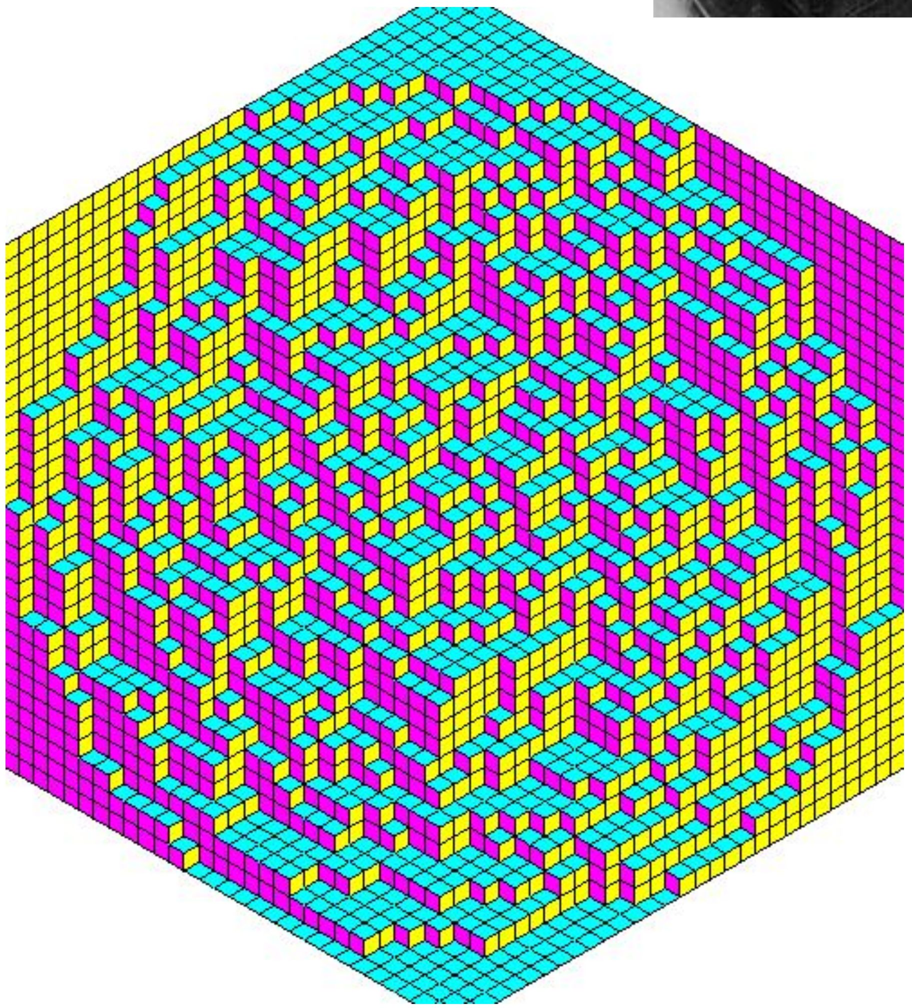
$$= 1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 11z^6 + 15z^7 + \ldots$$

Formula for q(n)

1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 18, 22, 27, 32, 38, 46, 54, 64,…

Plane Partitions





$$\sum_{n=0}^{\infty} \mathrm{PL}(n) x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + \cdots$$

Appendix

p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(11)

{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}