BMC Intermediate II — October 5, 2022

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Notation: For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n including 1 and n. Sometimes $d(n)$ is also used.

Miscellaneous number theory problems for beginners

1 Find the smallest integer greater than 1 which has a remainder of 1 upon division by $2,3,4,56,7,8,9,10$. Find the smallest positive integer which has a remainder of $1,2,3,\ldots,9$ when divided by $2,3,\ldots,10$, respectively.

2 Lockers in a row are numbered $1,2,3,\ldots,1000$. At first, all the lockers are closed. A person walks by and opens every other locker, starting with locker #2. Thus lockers $2,4,6,\ldots,998,1000$ are open. Another person walks by, and changes the "state" (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, starting with locker #3. Then another person changes the state of every fourth locker, starting with #4, etc. This process continues until no more lockers can be altered. Which lockers will be closed?

3 Show that if $a^2+b^2=c^2$, then $3|ab$.

4 If $x^3+y^3=z^3$, show that one of the three must be a multiple of 7.

5 Make sure that you know why 100! ends in 24 zeros and 1000! ends in 249 zeros. Can $n!$ end with $n/4$ zeros?

6 Find the smallest positive integer n such that $\tau(n)=10$.

7 Find the remainder when 2^{1000} is divided by 13.

8 Define the "repunit" R_n to be the number consisting of n consecutive 1s. For example, $R_5=11111$. Suppose R_n is prime? What can you say about n?

9 Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

10 What kind of numbers can be written as the sum of two or more consecutive integers? For example, 10 is such a number, because $10=1+2+3+4$. Likewise, $13=6+7$ also works.

11 *BAMM 2002*. Each of the following are products of two primes. Only one of these products can be written as the sum of the cubes of two positive integers. Which one?

(A) 104729×8512481779 (B) 104729×8242254443 (C) 104761×8242254443
(D) 104761×11401596337 (E) 104729×11401596337

12 Twin primes are pairs of prime numbers that are consecutive odd numbers, such as 17 and 19, or 41 and 43. The product of a pair of twin primes equals 55206201D99, where the third-from-last digit is the value D. Find D.

13 A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

14 How many ordered pairs (x, y) of integers are solutions to

$$\frac{xy}{x+y} = 99?$$

15 Find all positive integer solutions (x, y, z) to $105^x + 211^y = 106^z$.

16 Let $f(n)$ denote the sum of the digits of n. Let $N = 4444^{4444}$. Find $f(f(f(n)))$, without a calculator.

17 Find the last three digits of 7^{9999}.

18 Let $\{a_n\}_{n \geq 0}$ be a sequence of integers satisfying $a_{n+1} = 2a_n + 1$. Is there an a_0 so that the sequence consists entirely of prime numbers?

19 Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1{,}599.$$