Notation: For positive integers \( n \), let \( \tau(n) \) denote the number of positive integer divisors of \( n \) including 1 and \( n \). Sometimes \( d(n) \) is also used.

**Miscellaneous number theory problems for beginners**

1. Find the smallest integer greater than 1 which has a remainder of 1 upon division by 2, 3, 4, 5, 6, 7, 8, 9, 10.

2. Find the smallest positive integer which has a remainder of 1, 2, 3, ..., 9 when divided by 2, 3, ..., 10, respectively.

2. Lockers in a row are numbered 1, 2, 3, ..., 1000. At first, all the lockers are closed. A person walks by and opens every other locker, starting with locker #2. Thus lockers 2, 4, 6, ..., 998, 1000 are open. Another person walks by, and changes the “state” (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, starting with #3. Then another person changes the state of every fourth locker, starting with #4, etc. This process continues until no more lockers can be altered. Which lockers will be closed?

3. Show that if \( a^2 + b^2 = c^2 \), then \( 3 | ab \).

4. If \( x^3 + y^3 = z^3 \), show that one of the three must be a multiple of 7.

5. Make sure that you know why 100! ends in 24 zeros and 1000! ends in 249 zeros. Can \( n! \) end with \( n/4 \) zeros?

6. Find the smallest positive integer \( n \) such that \( \tau(n) = 10 \).

7. Find the remainder when \( 2^{1000} \) is divided by 13.

8. Define the “repunit” \( R_n \) to be the number consisting of \( n \) consecutive 1s. For example, \( R_5 = 11111 \). Suppose \( R_n \) is prime? What can you say about \( n \)?

9. Let \( P \) be the product of the first 100 positive odd integers. Find the largest integer \( k \) such that \( P \) is divisible by \( 3^k \).

10. What kind of numbers can be written as the sum of two or more consecutive integers? For example, 10 is such a number, because 10 = 1 + 2 + 3 + 4. Likewise, 13 = 6 + 7 also works.

11. **BAMM 2002.** Each of the following are products of two primes. Only one of these products can be written as the sum of the cubes of two positive integers. Which one?

    - A. \( 104729 \times 8512481779 \)
    - B. \( 104729 \times 8242254443 \)
    - C. \( 104761 \times 8242254443 \)
    - D. \( 104761 \times 11401596337 \)
    - E. \( 104729 \times 11401596337 \)
12 Twin primes are pairs of prime numbers that are consecutive odd numbers, such as 17 and 19, or 41 and 43. The product of a pair of twin primes equals 55206201D99, where the third-from-last digit is the value $D$. Find $D$.

13 A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

14 How many ordered pairs $(x, y)$ of integers are solutions to \[ \frac{xy}{x + y} = 99? \]

15 Find all positive integer solutions $(x, y, z)$ to $105^x + 211^y = 106^z$.

16 Let $f(n)$ denote the sum of the digits of $n$. Let $N = 44444444$. Find $f(f(f(n)))$, without a calculator.

17 Find the last three digits of $7^{9999}$.

18 Let $\{a_n\}_{n \geq 0}$ be a sequence of integers satisfying $a_{n+1} = 2a_n + 1$. Is there an $a_0$ so that the sequence consists entirely of prime numbers?

19 Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to \[ n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1,599. \]