Exponential Functions

D. Laws of Exponents

$$b^m \cdot b^n = b^{m+n}$$ $2^2 \cdot 2^3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$

$$\frac{b^m}{b^n} = b^{m-n}$$

$$(b^m)^n = b^{m \cdot n}$$

$$(a \cdot b)^m = a^m \cdot b^m$$

$$b^0 = b^{m-m} = \frac{b^m}{b^m} = 1 \Rightarrow b^0 = 1$$

$$b^{-m} = b^{0-m} = \frac{b^0}{b^m} = \frac{1}{b^m}$$

$$b^{m/n} = \sqrt[n]{b^m}$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

exercise

Give a numerical example to show that the following statements are false.

a) $(a+b)^m = a^m + b^m$

$a=1 \quad b=1 \quad m=2$

$(2)^2 = 1^2 + 1^2$

$4 \neq 2$

b) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

$a=16 \quad b=9$

$\sqrt{25} = \sqrt{16} + \sqrt{9}$

$5 \neq 7$

$\dfrac{1}{\sqrt{a+b}}$ $\Bigg\{$ $\dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

D2 Exponential Functions

$f(x) = a \cdot b^x$

$a \neq 0$, b is positive, $b \neq 1$

→ a is our initial value

$f(0) = a$

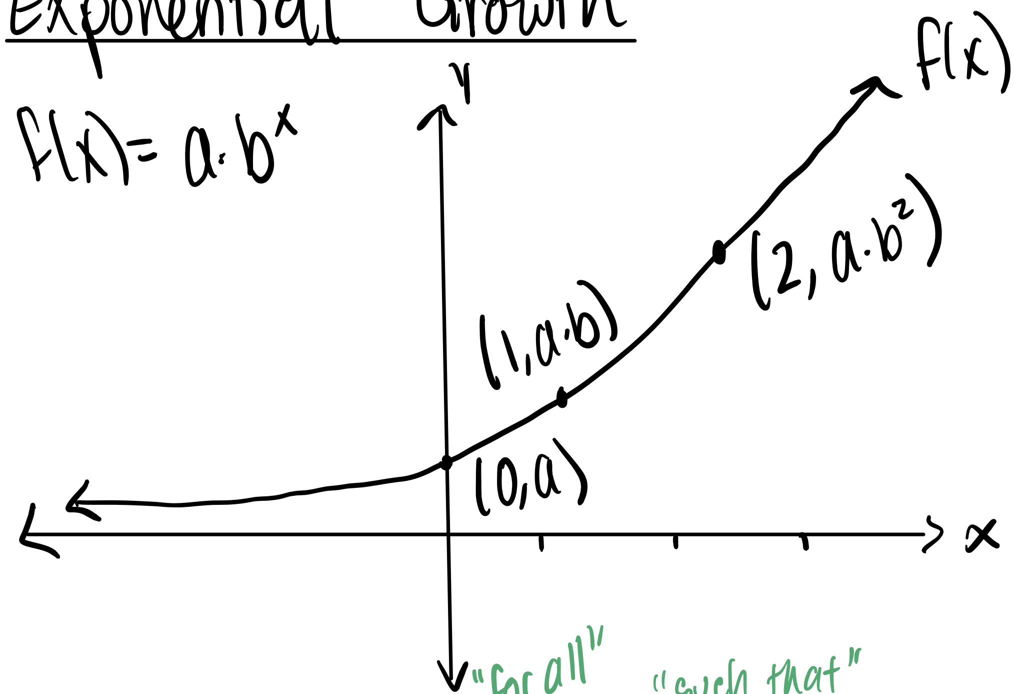
→ b is the "base": grow-by value

$0 < b < 1$ decay

$b > 1$ growth

Exponential Growth

$f(x) = a \cdot b^x$



"for all" "such that" set of real numbers

$\mathbb{R}$ $\mathbb{Z}$ $\mathbb{C}$ $\mathbb{N}$

- Domain: $\{\forall x \mid x \in \mathbb{R}\}$

  $\{x \mid x \in \mathbb{R}\}$ $x \in (-\infty, \infty)$

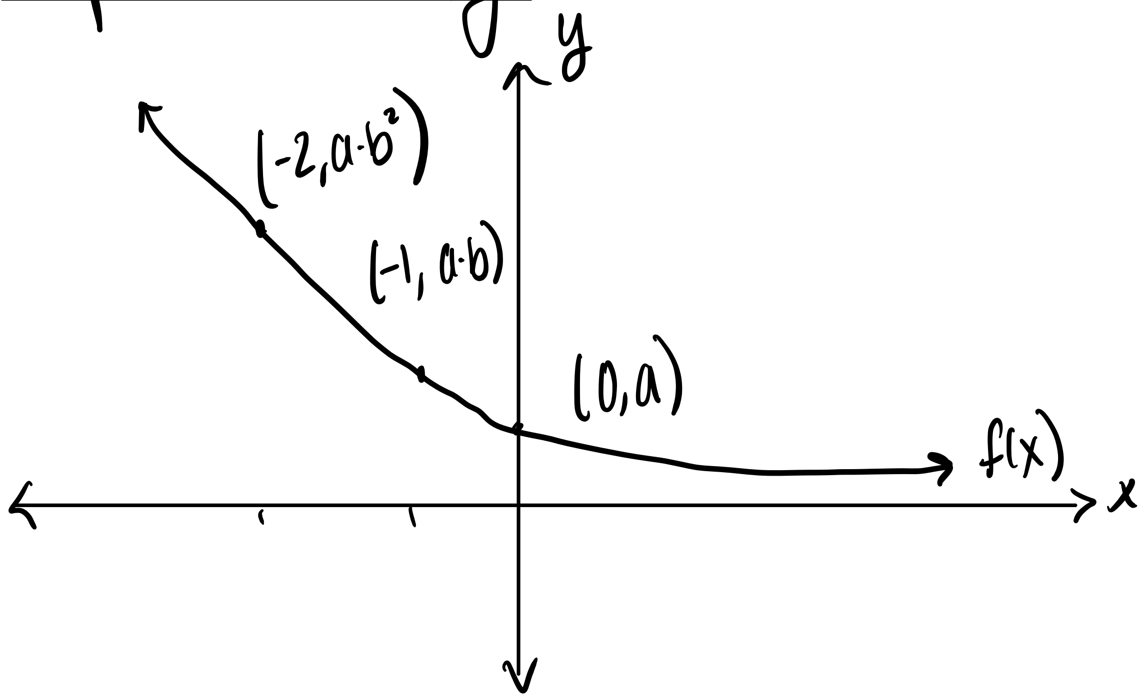
  $\in$ "an element of"

- Range: $\{y \mid y \in (0, \infty)\}$

  * $a > 0$ $b > 1$

- No vertical asymptotes
- Horizontal asymptote $y = 0$
- Increasing everywhere on the domain

Exponential Decay



- Domain: $\{x \mid x \in \mathbb{R}\}$
- Range: $\{y \mid y \in (0, \infty)\}$

  * $a > 0$ $\quad 0 < b < 1$

- No vertical asymptote
- Horizontal asymptote at $y = 0$
- Function is decreasing everywhere

D3 the natural base e

"e" Euler's number

$e \approx 2.71828...$ ← irrational number

we can approximate e using

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Thm. Any exponential function

$f(x) = a \cdot b^x$ can be rewritten

as $g(x) = a \cdot e^{k \cdot x}$ $\quad b = e^k$

→ $k > 0$ exp growth
→ $k < 0$ exp decay

D4 Population growth model

if a population is changing at a constant % rate r /year, then

$$p(t) = P_0 (1+r)^t$$

P_0 initial population
r decimal rep. of % growth
t time (years, months, min, etc)

$\boxed{D_5}$ Interest

$$a(t) = P_0 (1+r)^t$$

compounding yearly

$$b(t) = P_0 \left(1 + \frac{r}{n}\right)^{t \cdot n}$$

→ n # of times we compound per year

$$c(t) = P_0 \cdot e^{rt}$$

compound continuously

<u>exercise</u>

What is the difference between

$x^5 = 70$ $\qquad$ $5^x = 70$

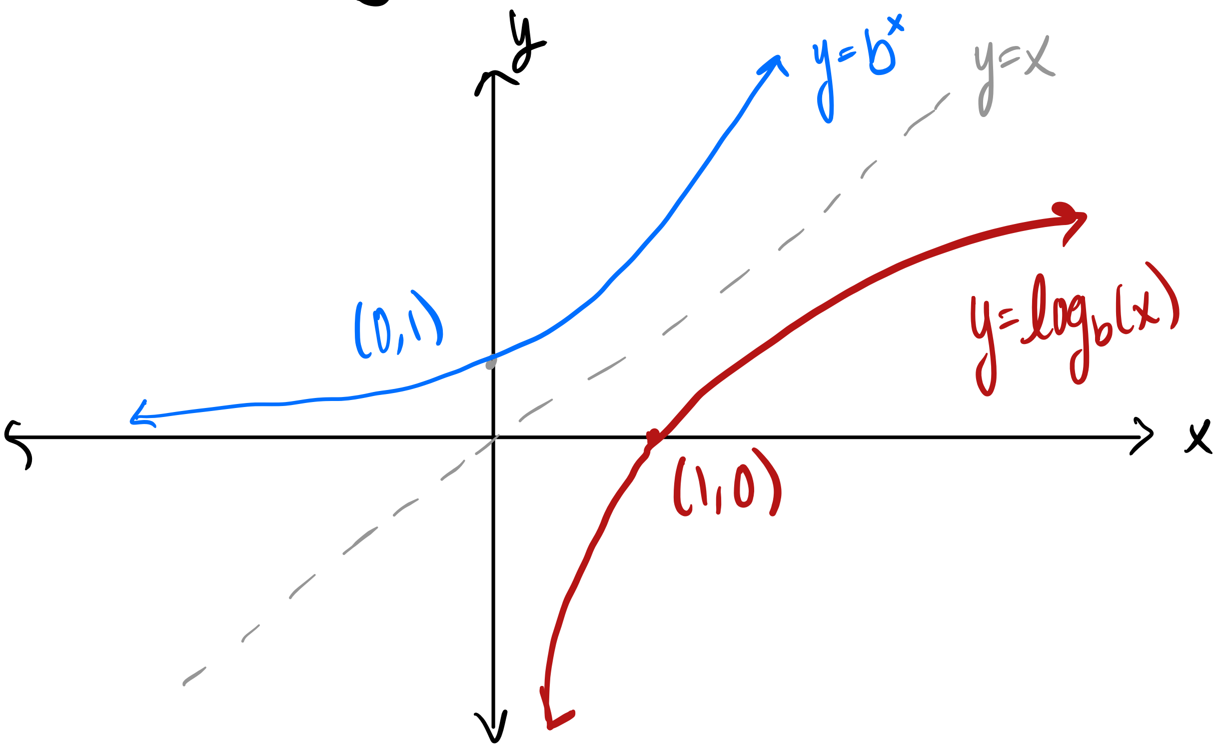
$x = \sqrt[5]{70}$ $\qquad$ $x = \log_5(70)$

$\boxed{D_6}$ **Logarithmic Functions**

If $x>0$, $b>0$ $b\neq 0$ then

$y = \log_b(x)$ if and only if

$b^y = x$



- Domain (Range) $\{x \mid x \in (0,\infty)\}$

- Range $\{y \mid y \in \mathbb{R}\}$
- increasing everywhere
- Vertical asymptote at $x=0$
- No Horizontal asymptote

D7 Laws of logs

* $b>0$ $b \neq 1$ $x>0$ any real y

- $\log_b(1)=0$ $b^0=1$
- $\log_b(b)=1$ $b^1=b$
- $\log_b(b^y)=y$ $b^y=b^y$
- $b^{\log_b(x)}=x$ $\sqrt{x^2}=x$

⚠ $\log_{10}(a)$ $\ln(a)=\log_e(a)$ "natural log"

- $\log(a \cdot b)=\log(a)+\log(b)$

WTS: $\log_b(M \cdot N) = \log_b(M) + \log_b(N)$

<u>Proof</u>

Let $M = b^x$ $N = b^y$ $x, y \in \mathbb{R}$

then by definition, it is also true that $\log_b(M) = x$ $\log_b(N) = y$

$LHS = \log_b(M \cdot N)$

$= \log_b(b^x \cdot b^y)$

$= \log_b(b^{x+y})$

$= x + y$

$= \log_b(M) + \log_b(N)$ $\square$

- $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

- $\log(a^n) = n \cdot \log(a)$

- $\log_n(a) = \log(a)$ "change of

$\log(b)$ "base" formula"

- $\log\left(\frac{1}{a}\right) = \log(1) - \log(a)$

$= 0 - \log(a) = -\log(a)$

$\hookrightarrow \log(a^{-1}) = -\log(a)$

Solve for x algebraically

① $5^{2x+4} = 5^{x^2}$

② $5^{4-3x} = 25^{x+2}$

③ $4(125)^x = 30(5^{2x})$

④ $7e^{-3x} = 63$

⑤ $2e^x = 50e^{-2x}$

6. $\log_2(x^2+1) = 10$

7. $3\log(x) = \log(16) - \log(2)$

8. $4\ln(x) - 3\ln(2) + \ln(16) = 20$