23rd Bay Area Mathematical Olympiad

BAMO-12 Exam

March 2, 2022

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument that has a few minor errors may receive substantial credit.

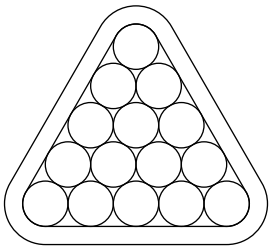
Please label all pages that you submit for grading in the following way: each new problem should start with a title page (which you should have received). Write your identification number in the box on the left and write your identification number *again* in the box on the right, but with the letter "A" at the beginning. For example, if your id # is 31416, then you write **31416** in the left box and **A31416** in the right box. (Your papers will be scanned and these duplicate ID #s will help the grading platform recognize your ID #.) If your work uses extra pages, use plain blank white paper and wrote your the problem name, your ID# and what page it is on the bottom of the page. For example, write "Problem A, p. 2 of 3, ID 31416."

Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Problems 1, 2, and 3 are on this side; problems 4 and 5 are on the other side.

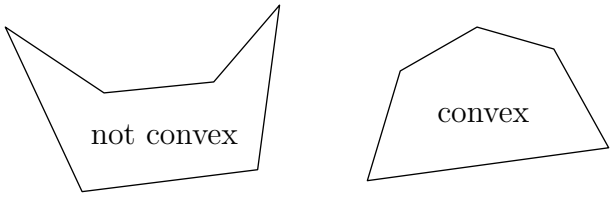
1 The game of pool includes 15 balls that fit within a triangular rack as shown:



Seven of the balls are striped. Prove that no matter how the 15 balls are arranged in the rack, there must always be a pair of striped balls adjacent to each other.

2 Suppose that p, $p+d$, $p+2d$, $p+3d$, $p+4d$, and $p+5d$ are six prime numbers, where p and d are positive integers. Show that d must be divisible by 2, 3, and 5.

3 A polygon is called *convex* if all of its internal angles are smaller than 180°. Here are examples of nonconvex and convex polygons:



Given a convex polygon, prove that one can find three distinct vertices A, P, and Q, where PQ is a side of the polygon, such that the perpendicular from A to the line PQ meets the segment PQ (possibly at P or Q).

4 Ten birds land on a 10-meter-long wire, each at a random point chosen uniformly along the wire. (That is, if we pick out any x-meter portion of the wire, there is an $\frac{x}{10}$ probability that a given bird will land there.)

What is the probability that every bird sits more than one meter away from its closest neighbor?

5 Sofiya and Marquis are playing a game. Sofiya announces to Marquis that she's thinking of a polynomial of the form $f(x) = x^3 + px + q$. She also explains that her polynomial has three integer roots, not necessarily distinct, all of which have absolute value less than (and not equal to) N, where N is some fixed number which she tells Marquis. As a "move" in this game, Marquis can ask Sofiya about any number x and Sofiya will tell him whether $f(x)$ is positive, negative, or zero. Marquis's goal is to figure out Sofiya's polynomial.

If $N = 3 \cdot 2^k$ for some positive integer k, prove that there is a strategy which allows Marquis to identify the polynomial after making at most $2k+1$ "moves".

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2022 Awards Ceremony**, which will be held at Santa Clara University in the afternoon on Sunday, March 20. This event will include a mathematical talk by **Alon Amit (Intuit and Quora)**, and the awarding of prizes. Please check with your proctor and/or `https://www.bamo.org` for a more detailed schedule, plus directions.