Hat Problems I

BMC Adv Fall 2022

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1 Warm Up

1. A king blindfolds three ministers and places a white or black hat on each of them. He then removes the blind folds and tells them to raise their hand if they can see a black hat. They all raise their hands. Then he tells them to keep their hand up if they know what color hat they have on. They all lower their hands, and then one of the ministers raises his hand. What color is his hat?

2. This time, the king has the three ministers brought before him as he blindfolds them and puts a white or black hat on each. He then places them in line facing forward and takes off their blindfolds, so the person in back can see two hats and the person in front cannot see anything. He tells them that there is at least one hat of each color. He now asks them if they know what color hat they have. No one says anything and then someone says that they know their hat color. Who said it and how?

3. The king now brings ten ministers brought before him as he blindfolds them and puts a white or black hat on each of them. He again places them in line facing one direction and takes off their blindfolds. He starts in the back and asks them in order what color hat they have, and they can hear what the ministers behind them have said. He tells them if they all answer correctly, he will give them each a million dollars. What should be their strategy to maximize their probability of winning?

4. This time with the ten ministers, the king lets them stand in a circle so that they can see everyone else's hats. Now however, he tells them that they must all guess what color hat they have at the same time. He tells them if at least five of them answer correctly, he will give each of them a million dollars. What should be their strategy to maximize their probability of winning?

5. The king now brings all the ministers of his kingdom in and has them blindfolded and places a reversible hat on each. The hat can be turned inside out from white to black or vice versa, and its initial orientation is random. He tells them that he sees 10 white hats and keeps them blindfolded and tells them to divide themselves into two groups so that each group has the same number of people with white hats. How can they do this?

2 Error-Correcting Codes

1. When we send data of the internet, we encode everything in terms of 0s and 1s. One simple way to do this is by letting a be 00001 for 1 and z be 11010 for 26. How would you send "hello"?

2. Often times, data can be sent incorrectly over the internet and we need to know whether the data we received is correct. One easy way to do this is by adding a parity bit which is the sum of all the digits modulo 2. For instance, a would now be 000011 and z would be 110101. Using this, how would you send "hello"? In real life, this is used for credit cards as the last digit of a credit card number is a parity check.

3. What kind of errors can the parity bit identify? Is it possible to figure out what the original message was if we know there was only one error?

4. If we want to send either a 0 or a 1, what is one way to send it so that even if there is a single error, the receiver would know what we sent?

5. The Hamming(7, 4) code uses 7 bits to send 4 original bits and is such that any single bit error can be detected and fixed. It is given by
$$p_1p_2d_1p_3d_2d_3d_4$$
where p_i are the parity bits and d_i are the original message bits. We have that $p_1 = d_1+d_2+d_4, p_2 = d_1 + d_3 + d_4, p_3 = d_2 + d_3 + d_4$. How would we send "hello"? (ignore the leading 0).

6. The message 1011011 has an error. What is the original message? The message 1010011 has an error. What is the original message? The message 1110111 has an error. What is the original message?

3 Ebert's Hat Problem

1. The king brings the original three ministers before him and blindfolds them and places a white or black hat on each. He removes their blindfolds so they can see the other hats and tells them that on the count of three they must either stay silent or guess what color hat they have. If at least one of them guesses the color of their own hat and no incorrect guesses are made, they each get a million dollars. What should be their strategy to maximize their probability of winning?

2. The king now brings in seven ministers and does the same thing. What should be their strategy to maximize their probability of winning?

3. The king now brings in four ministers and does the same thing. What should be their strategy to maximize their probability of winning?

4. Prove that if there are n ministers, then their probability of winning is at most $\frac{n}{n+1}$. Optimal strategies are known for $n = 2, 3, 4, 5, 6, 7, 8, 2^k - 1$ people.

5. If we have the original three ministers but now everyone must guess (they cannot remain silent), what should be their strategy if they want a majority of them to answer correctly?