Puzzles in combinatorics and probability

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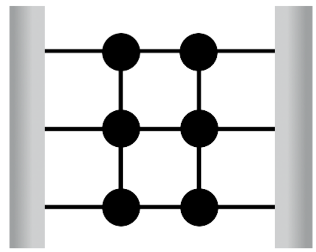
1. There are 1000 green balls and 3000 red balls in container A, and 3000 green balls and 1000 red balls in container B. You take half of the balls from A at random and transfer them to B. Then you take one ball from B at random. What is the probability that this ball is green?

2. Six islands are connected with two banks of the river with 13 bridges as shown in the picture below.



   When the flood occurs each bridge is independently destroyed with probability $\frac{1}{2}$. What is the probability that it will be possible to cross the river after the flood using the bridges that remain?

3. Two rooms A and B initially contain 1 person. Each second a new person arrives to one of the rooms. If there are a people in A and b people in B the person arriving will choose to go to room A with probability $\frac{a}{a+b}$ or to room B with probability $\frac{b}{a+b}$.

   Determine the distribution of the random variable $\min\{A_n, B_n\}$ where A_n is the number of people in the room A and B_n is the number of people in the room B after n seconds.

4. Evil Commander took the cell-phones from all of his one hundred soldiers. He then correctly wrote the names of soldiers on the phones, but intentionally placed phones randomly in boxes labeled by 1, 2, ..., 100. One by one the soldiers are taken to the room with boxes. Once in the room, a soldier is allowed to perform the following 3-step procedure at most 50 times:

   - Step 1. Choose one of the boxes;
   - Step 2. Open the box;
   - Step 3. If the box contains the soldier's own cell-phone, the soldier uses the fingerprint technology to unlock it. Then he can send a message to the President voicing the discontent with Evil Commander.

   After repeating the procedure at most 50 times, the soldier must close all boxes and leave the room without taking any phones regardless whether the soldier succeededs in finding his/her own device.

   However, if the President receives 100 messages (one from each soldier), then the President will force the Evil Commander to return the phones to the soldiers. However, if at least one of the soldiers fails to find the phone in 50 attempts or fewer, then the President will believe to Evil Commander who will deny any mischief and none of the soldiers will get the phone back.

   The night before the game starts, the soldiers are allowed to discuss and make the strategy. Prove that there is a strategy that results in success with probability more than $\frac{1}{4}$.

5. Seven dwarfs are captured by the evil queen who decided to play the following game. The queen puts a red hat or a green hat on the head of each of the dwarfs. The hats are chosen randomly and every configuration is equally likely. The dwarfs can see all the hats except for their own.

   At a signal each dwarf can stay silent, or guess the color of his hat. The queen will free all seven dwarfs if at least one dwarf guesses his hat correctly and no one guesses the hat incorrectly. If all the dwarfs are silent, or some dwarfs say incorrect color, then the queen cooks and eats all of the dwarfs.

   Before the game starts, the dwarfs could decide on a strategy. Prove that there is a strategy that can result in freedom with probability higher than 85%.

6. Starting with an empty $1 \times n$ board (a row of n squares), we successively place 1×2 dominoes to cover two adjacent squares. At each stage, the placement of the new domino is chosen at random, with all available pairs of adjacent empty squares being equally likely. The process continues until no further dominoes can be placed. Find the limit, as n goes to infinity, of the expected fraction of the board that is covered when the process ends.

7. A player chooses a natural number k smaller than or equal to 52. The top k cards are drawn one by one from a properly shuffled standard deck of 52 cards. The player wins if the last drawn card is an Ace, and if there is exactly one more Ace among the cards drawn. Which k should the player choose to maximize the chance of winning in this game?

8. A box contains 100 green, 100 white, and one red ball. A player draws balls from the box without replacement and earns \$1 for each green ball and no money for white balls. The game is over once the player chooses to stop or once the red ball is drawn. If the game ends with the red ball, the player looses all the money. However, if the game ends because the player chose to stop, the player can keep the money. What is the best strategy that the player should use to maximize the gain?

9. A standard deck of 52 playing cards is shuffled and cards are flipped over, in sequence, one at a time. Immediately before each flip, you have the opportunity to bet any amount of money that you have, from \$0 to everything you have, on the color of the card that the dealer is about to flip. So, for instance, if you have \$5 and the dealer is about to flip a card, you may either do nothing, bet any amount of money up to \$5 that the card will be red, or bet any amount of money up to \$5 that the card will be black. A correct bet of \$$x$ wins you \$$x$; an incorrect bet costs you \$$x$. You begin the game with \$100. At any point in the game, you can recall perfectly the sequence of cards that has been flipped. Assume that dollars are continuously divisible. That is, whenever you chose to bet, you can bet any positive real number of dollars. What is the maximum amount of money you can be guaranteed to have once the deck is through, and what betting strategy should you use to achieve this outcome?

10. What is the probability that two randomly chosen positive integers are relatively prime?