The 15-Puzzle Puzzled Out

with Zvezdelina Stankova
Teaching Professor of Mathematics
University of California at Berkeley
Founder and Director of the Berkeley Math Circle at UCB
Founding Director of the Math Taught the Right Way at UCB

You see below a 3x3 image of Cindy Lawrence, our host, made of 9 squares. But something's wrong! Two squares in the rightmost column are flip-flopped! Can we correct this and see the full picture of Cindy?

To the rescue comes the bottom left corner, which is empty and you can slide adjacent squares into the empty slot in trying to arrange the puzzle. If you are successful, the puzzle will fill in the empty square, giving the full picture:

https://www.geogebra.org/m/yqczqvxy

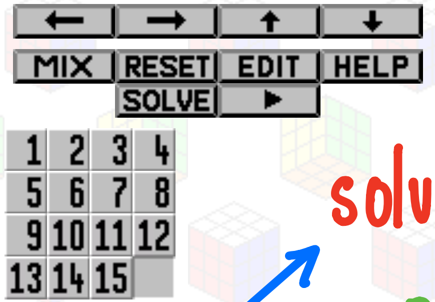
The original and most popular version of this puzzle is the so-called 15-puzzle, made of 16 squares arranged in a 4x4 table and labeled 1-15. One square is empty and you can use it to try to arrange the puzzle in order. However, which mixed arrangements are possible to solve and which are not? In other words, if you take the puzzle apart and randomly put it back together, what is the chance that it will be solvable?

While the chances here will be fairly high: 50-50, you will not be so lucky with the Rubic's cube: only 1 in 12 randomly assembled versions will be solvable. Yet the explanation for the 15-puzzle and the Rubic's cube are of the same flavor and use deep ideas from Group Theory: a must to explore by any game fan and math aficionado.

In this talk, we will concentrate on demystifying the 15-puzzle, both in practice and in theory, and learn to immediately catch if anyone has cheated (by taking it apart and putting it back together) and has given us a "defective" puzzle! If you would like to get a head-start, try various puzzle sizes at

https://www.jaapsch.net/puzzles/javascript/fifteenj.htm
by Jaap Scherphuis

The Plan



Part III

15-puzzle
- solving & beyond!
- possible vs. impossible states
- closed paths
- as action group

Part II

Permutation Groups
- fund'l theorems: even vs. odd permutations
- products of transpositions
- 2-row notation ✓
- 1-row (cyclic) notation ✓

Part I ✓

Groups
- examples: ✓ action groups
- intro ✓ definition
- bits of theory & group "jargon" ✓

Part III

Action Groups (Lecture)

Worksheet 5: Alternating Group and the 15-Puzzle Puzzled Out[1]

Date: 11/20/2020

MATH 74: Transition to Upper-Division Mathematics
with Professor Zvezdelina Stankova, UC Berkeley

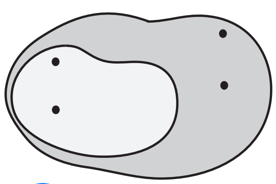
Read: *Session 5: Introduction to Group Theory.* (vol. II, pp. 120-125)

- §6.4. Permutation are born unequal
- §7. The 15-Puzzle Puzzled Out

Write: clearly. Supply your reasoning in words and/or symbols. Show calculations and relevant pictures.

1. **(No Double-Dipping!)** Prove the fundamental facts about even and odd permutations in S_n:
   (a) The identity permutation is *not* odd.
   (b) Every $\alpha \in S_n$ is even or odd but *not both*.
   (c) An r-cycle is even if and only if r is odd.
   (d) A permutation is even iff its cyclic decompositions have even number of even-length cycles.

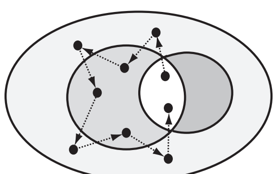
2. **(Groupy Evens)** Prove that:
   (a) If $\alpha, \beta \in S_n$ are even, then $\alpha \circ \beta$ and α^{-1} are also even permutations in S_n.
   (b) The set A_n of all even permutations in S_n is a *subgroup* of S_n but the set of odd ones is not.



1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

4	3	2	1
5	6	7	8
12	11	10	9
13	14	15	16

3. **(Alternating Group)** Consider the alternating group A_4 of even permutations on 4 elements.
   (a) What is $\circ(A_4)$? Explain.
   (b) List all elements of A_4.



10	9	8	7
11	2	1	6
12	3	4	5
13	14	15	16

8	14	11	3
12	2	15	9
6	4	13	1
7	10	5	16

4. **(Set-Up for 15-Puzzle)** Imagine that the number 16 is written in the empty cell. Show that:
   (a) We can interpret all states of the 15-puzzles as permutations in S_{16}. (*Hint*: Each move is a transposition of 16 with an adjacent square.)
   (b) The 15-puzzle arrangements on this page be interpreted as permutations in S_{15}, but that will not be helping in solving the puzzle.
   (c) The first arrangement is the identity (1); the second is a product of 4 transpositions; the third is an 11-cycle starting (1, 10, 3, ...). Write the fourth in cyclic notation.
   (d) Decide which of these four arrangements are odd and which are even permutations.

5. **(Moving in Puzzleland)** Prove that any sequence of moves in the 15-puzzle can be:
   (a) viewed as a path in S_{16} that alternates between even and odd permutations.
   (b) written as a product of transpositions with 16: $(a_1, 16)(a_2, 16) \ldots (a_k, 16)$.

6. **(Impossible States)** Prove that no odd permutation (of S_{16}) can be reached in the 15-puzzle. Which of the given arrangements are impossible?

7. **(Possible States)** Prove that any even permutation can be obtained in the 15-puzzle. Start with an arrangement and reverse the process to reach the identity (Problem 18, p. 125):
   (a) Arrange consecutively rows 1, 2, 3, without touching the rows above, and push the empty cell to the right bottom position.
   (b) If row 4 shows any of the three sequences:
      - $\{13, 15, 14\}$, $\{14, 13, 15\}$, $\{15, 14, 13\}$ (transpositions), the puzzle is defective: the original permutation is odd and not reachable.
      - $\{13, 14, 15\}$, $\{14, 15, 13\}$, $\{15, 13, 14\}$ (even, 3-cycles), we go on. The first one is the identity: done! Since $(15, 13, 14) = (14, 15, 13)^2$, once we reach $\{14, 15, 13\}$ (see p. 140 for a way to do this by permuting only the bottom two rows), we apply again the same algorithm to reach $\{15, 13, 14\}$.

Thus, a permutation can be reached iff it is even.

[1]

① Reality Check: How does this work on 15-puzzle?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$e = (12)(12) \in S_{16}$

4	3	2	1
5	6	7	8
12	11	10	9
13	14	15	16

$\alpha \in S_{16}$

- abstractly:

$$\underbrace{(1,4)(2,3)(9,12)(10,11)}_{\text{disjoint cycles:}}$$

4 transpositions

- 2-row: $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 4 & 3 & 2 & 1 & 5 & 6 & 7 & 8 & 12 & 11 & 10 & 9 & 13 & 14 & 15 & 16 \end{pmatrix}$

10	9	8	7
11	2	1	6
12	3	4	5
13	14	15	16

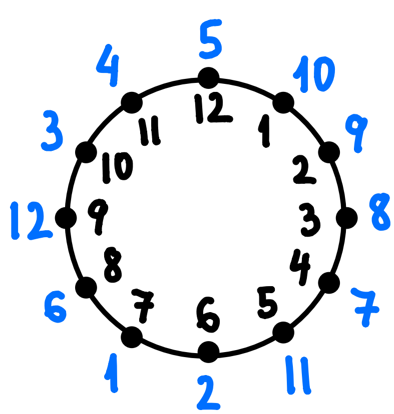
$$\begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 \\ 10, & 9, & 8, & 7, & 11, & 2, & 1, & 6, & 12, & 3, & 4, & 5, & 13, & 14, & 15, & 16 \end{pmatrix}$$

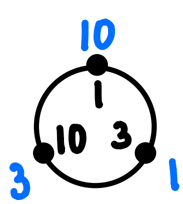
- 1-row: $\beta = (1,10,3,8,6,2,9,12,5,11,4,7)$

$$\boxed{\beta = (1,10)(10,3)(3,8)(8,6)(6,2)(2,9)(9,12)(12,5)(5,11)(11,4)(4,7)}$$

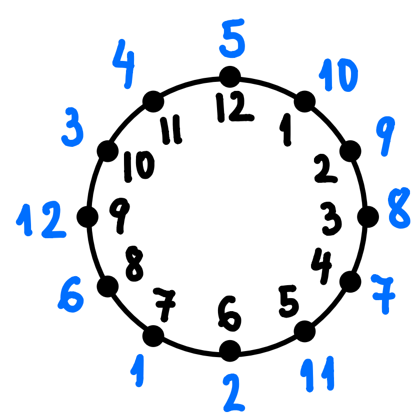
Another way: Ex $(1,10,3) = (1,3)(1,10)$

$$\boxed{\beta = (1,7)(1,4)(1,11)(1,5)(1,12)(1,9)(1,2)(1,6)(1,8)(1,3)(1,10)}$$





12-cycle: 11 transpositions



② Thm (Even/odd)

$\forall$ permutation $\alpha \in S_n$ can be written as a product of even or odd # of transpositions but Not both!

Examples:

- $e = (12)(12)$ even
- $(23) = (23)(34)(34)$ odd
- $(123) = (12)(23)(35)(35)$ even
- $\alpha = (1,4)(2,3)(9,12)(10,11)$ even
- $\beta = (1,10)(10,3)\cdots(4,7)$ odd

Ex. $(123) = (12)(23) \rightsquigarrow (312) = (31)(12)$ Not unique!

$(5781) = (57)(78)(81)$

$(2753)(14)(689) = (27)(75)(53)(14)(68)(89)$

Universal Algorithm:

any perm. α in S_n → 2-row notation "raw" → 1-row notation cycles → product of transp.

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 4 & 6 & 7 & 1 & 2 \end{pmatrix} \to (1346)(257) \to (13)(34)(46)(25)(57)$

disjoint cycles   non-disj. transp.

↓

number of transp. → type even/odd perm. → solvable or unsolv.

α: 5 transp. α: odd   ?

3.5 Odd or Even Cycle?

Lem The cycle $(a_1 a_2 \ldots a_r)$ is even iff r is odd.

"Pf": $(25134) \rightarrow$ length 5

$= \underbrace{(25)(51)(13)(34)}_{4 \text{ transp.}} \Rightarrow$ even ☺

3.9 Def:

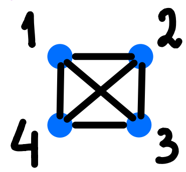
A_n = all even perm's in S_n

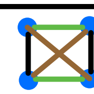
O_n = all odd perm's in S_n

Ex. $S_2 = \{e, (12)\}$ $\rightarrow A_2 = \{e\}$, $\rightarrow O_2 = \{(12)\}$

Ex. $S_3 = \{e, (123), (132),$ $\rightarrow 3$
$(12), (23), (13)\}$ $\rightarrow 3$

Ex. S_4: acting on $\{1,2,3,4\}$



O_4 = odd perm's		Tot: 12	A_4 = even perm's		Tot: 12
(ab)	transpositions (2-cycles)	$\binom{4}{2} = 6$	e	identity	1
$(abcd)$	4-cycles $(ab)(bc)(cd)$	$3! = 6$	$(abc) \neq (acb)$	3-cycles	$\binom{4}{3} \cdot 2 = 4 \cdot 2 = 8$
			$(ab)(cd)$	pairings	 $= 3$

tennis doubles

Thm For $n \geq 2$, $\boxed{\#A_n = \#O_n = n!/2}$

$\Rightarrow$ as many even as odd permutations!

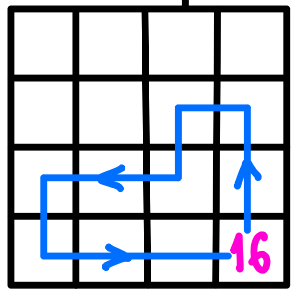
Q: Which are the "important" or "well-behaved" permutations?

④ Possible States:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$\xrightarrow[\text{w/ } 16]{\text{transp.}}$

Ex. 16-path



=

1	2	3	4
5	6	11	7
13	9	10	8
14	15	12	16

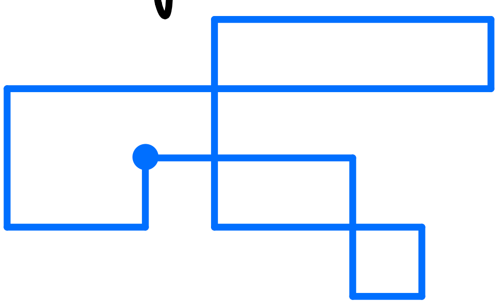
γ

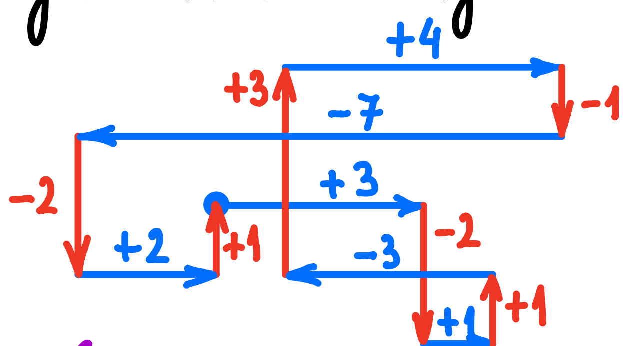
Q: What path did 16 trace?

$$\gamma = \left.\begin{matrix}(16,12)(16,15)(16,14)(16,13)(16,9) \\ (16,10)(16,11)(16,7)(16,8)(16,12)\end{matrix}\right\}\text{ 10 transp. even!}$$

Q: What kind of path? A: Closed!

Lem. Any closed path in a grid has even length.





$$\begin{cases} -2+1+3-1-2+1=0 \\ +3+1-3+4-7+2=0 \end{cases}$$

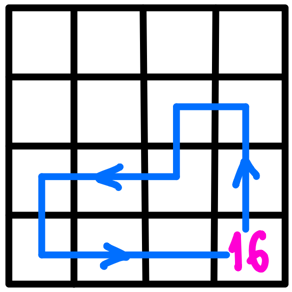
♡ **Moreover:**

\# hor. steps is even.

\# vert. steps is even.

$\#\leftarrow \;=\; \#\rightarrow \qquad \#\uparrow \;=\; \#\downarrow$

Ex. 16-path



\# hor. steps $= 3+3=6$
\# vert. steps $= 2+2=4$
$\Longrightarrow$ even permutation!

Cor: Any attainable state is a product of even # transp. (w/ 16), and hence "abstractly" an even perm.

Contrapositive:
No odd permutation is attainable!

10	9	8	7
11	2	1	6
12	3	4	5
13	14	15	16

odd ☹

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	16

(14,15): odd ☹

⑤ How to attain the attainable?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$\xrightarrow[\text{w/ } 16]{\text{? transp.}}$

8	14	11	3
12	2	15	9
6	4	13	1
7	10	5	16

even perm.
↑
length 15
↑

$\delta = (1,8,9,6,2,14,10,4,3,11,13,7,15,5,12) \in A_{16}$

♡ Still hopeful:
Q: Can we obtain δ?

♡ Create algorithm!

♡ Reverse task: attain $\underline{e}$ from any even perm.

Algorithm:

- Arrange
  - → row 1
  - → row 2 w/out touching row 1
  - → row 3 w/out touching rows 1 and 2

- Odd perm's: unobtainable!

1	2	3	4
5	6	7	8
9	10	11	12
14	13	15	16

$(13,14) \notin A_{16}$ ☹

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	16

$(14,15) \notin A_{16}$ ☹

1	2	3	4
5	6	7	8
9	10	11	12
15	14	13	16

$(13,15) \notin A_{16}$ ☹

- Even perm's: possible/solvable!

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$e \in A_{16}$ ✓

$\xrightarrow{\gamma}$

1	2	3	4
5	6	7	8
9	10	11	12
14	15	13	16

$\gamma = (13,14,15) \in A_{16}$ ✓

$\xrightarrow{\gamma}$

1	2	3	4
5	6	7	8
9	10	11	12
15	13	14	16

$\gamma^2 = (13,15,14) \in A_{16}$ ✓

$$\gamma^2 = (13,14,15)^2 = (13,14,15)(13,14,15) = (13,15,14) = \gamma^{-1}$$

- Q: How to obtain $\gamma = (13,14,15)$ from e?
  Equivalently, how arrange γ into e?

(1) Rotate the bottom two rows clockwise by one place.
(2) Rotate the bottom middle 2×2 square clockwise by one place.
(3) Rotate the bottom left 2×2 square counter-clockwise by one place.
(4) Move the square with 9 in it (to the left), and then move the square with 10 in it (up).
(5) Rotate the bottom two rows counter-clockwise by one place.

The last permutation is the desired identity permutation e. □

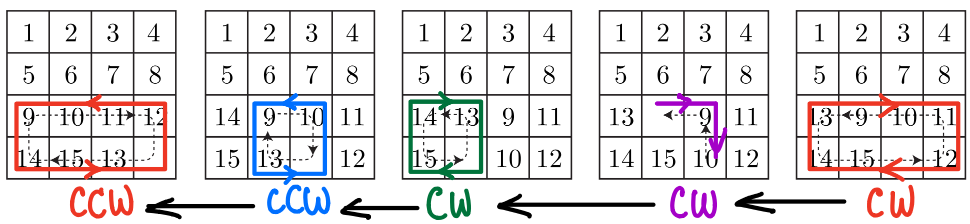


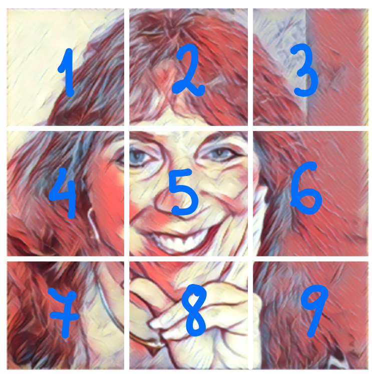
FIGURE 12. Permuting the 3-cycle $(13, 14, 15)$ to the identity e

The 15-Puzzle Puzzled Out

with Zvezdelina Stankova
Teaching Professor of Mathematics
University of California at Berkeley
Founder and Director of the Berkeley Math Circle at UCB
Founding Director of the Math Taught the Right Way at UCB

You see below a 3x3 image of Cindy Lawrence, our host, made of 9 squares. But something's wrong! Two squares in the rightmost column are flip-flopped! Can we correct this and see the full picture of Cindy?

To the rescue comes the bottom left corner, which is empty and you can slide adjacent squares into the empty slot in trying to arrange the puzzle. If you are successful, the puzzle will fill in the empty square, giving the full picture:



https://www.geogebra.org/m/yqczqvxy

$(6,7,9) \rightarrow$ 3-cycle

$= (6,7)(7,9) \Rightarrow$ even

$\Rightarrow$ solvable!

(hopefully! :))

While the chances here will be fairly high: 50-50, you will not be so lucky with the Rubic's cube: only 1 in 12 randomly assembled versions will be solvable. Yet the explanation for the 15-puzzle and the Rubic's cube are of the same flavor and use deep ideas from Group Theory: a must to explore by any game fan and math aficionado.

Q: What is left to solve? → n-puzzle; → Rubics cube + versions

Which and how many states are solvable? How to know if someone cheated?