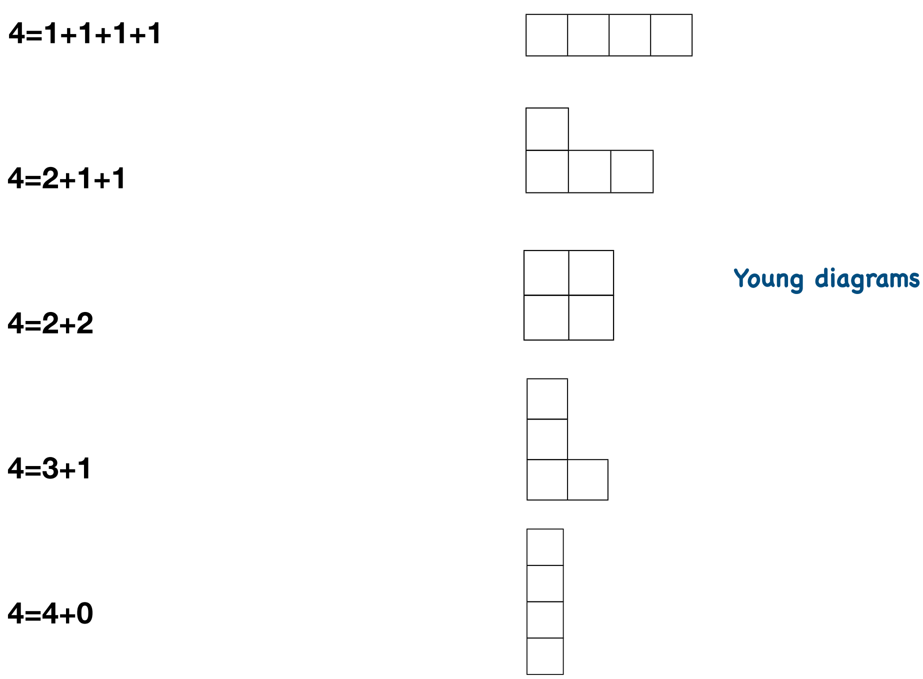
Integer Partitions

Contents

1. Partitions of Integers and Stacking Blocks

Given a positive integer, say 4, there are several different ways to decompose it into sums of positive integers which are not bigger than 4, see figure below:



On the left we see five possible decompositions of 4 and on the right – their visualizations in terms of so-called Young diagrams. The rules are the following:

- The number of blocks in each column of the diagram is equal to the corresponding summand, say on top of the figure $4 = 1 + 1 + 1 + 1$ has a row of four block as its Young diagram.

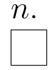
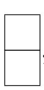
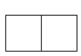
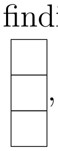
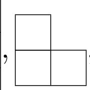
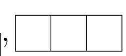
- In order to avoid double-counting and to count, say, $4 = 3 + 1$ and $4 = 1 + 3$ as the same partition we agree to place higher columns to the left and lower columns to the rights of the Young diagram. In other words, if we move from left to right the hight cannot increase.

In class we visualized partitioning of integers using toy blocks (we used some of my daughter's sets of cubic blocks). At the beginning start with the row partition, e.g $4 = 1 + 1 + 1 + 1$ or $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$. Then take the right-most block and move it to the left such that the above rules are satisfied. Sometimes there's only one option, sometimes there are several. Then take the next right-most (and top-most) block and move it to the left such that the rules are satisfied, etcetera. Keep doing this until you run out of options and no blocks can be moved to the left.

In the above figure the partitions (as well as the diagrams) are ordered according to the this algorithm – each diagram in the list can be obtained from one of the diagrams above it by moving a block from right to left.

Our first problem was to list all partitions and Young diagrams of integers 1 through 7. We used my daughter's blocks as well as simply grid paper and pencil – it is easy to visualize this process and draw all the steps:

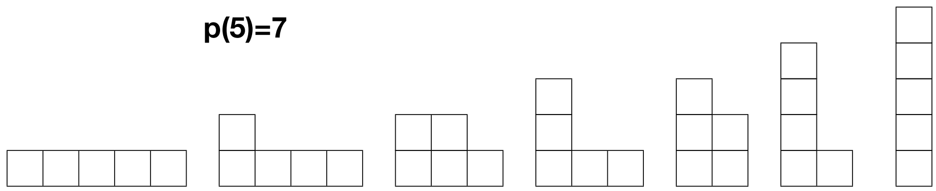
n	1	2	3	4	5	6	7
p(n)	1	2	3	5	7	11	15

Here n is the integer and $p(n)$ is the total number of partitions of n. The first three partitions are easy and kids had no trouble finding the only option  for $n = 1$, two options ,  for $n = 2$ and three options , ,  for $n = 3$.

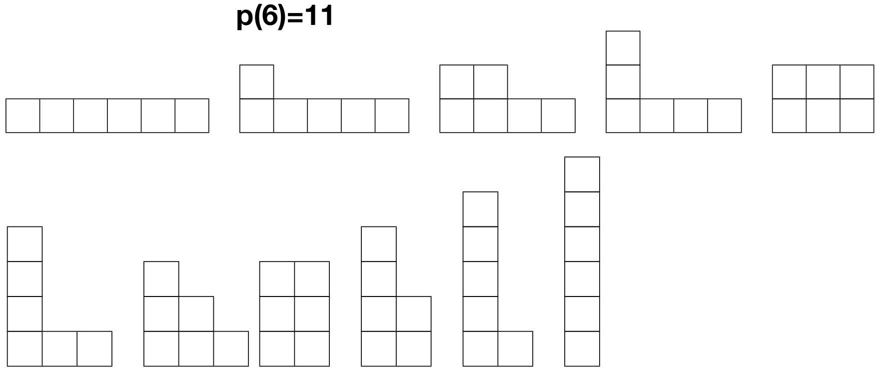
Interestingly, when the class was working on $n = 4$, some kids suggested that the sequence $p(n)$ should be the Fibonacci sequence which we studied in details last year. Unfortunately, after the number of partitions for $n = 5$ was computed, the hope to see Fibonacci again had faded away. In fact, the sequence of integer partitions is much more complicated than Fibonacci sequence and it is our goal for the next several classes to understand its symmetries.

Here are the Young diagrams for $n = 5$ and $n = 6$:

n=5 **p(5)=7**



n=6 **p(6)=11**



Some kids found all 15 partitions of 7, if you have not done it in class then **Problem:** complete the table for $n = 7$ at home.

Problem 1. Odd and distinct parts. We observed in class that the total number of odd partitions is equal to the total number of distinct partitions for each integer n. Try to understand why this is the case.

2. Odd and Distinct Parts

It is not clear at this moment what is the rule behind the sequence of partitions $p(n)$. Let us try instead to look at subclasses of partitions and understand if they form a simpler pattern.

Define *odd partition* to be a partition whose all parts are odd, i.e. $3 = 1 + 1 + 1$ (1 are odd), $4 = 3 + 1$ (3 and 1 are odd), etc. The partition $4 = 2 + 2$ is obviously not odd.

Additionally define *distinct partition* to be a partition whose parts are mutually distinct (no two parts are the same): $4 = 3 + 1$, $5 = 3 + 2$, $7 = 4 + 3$ are distinct, however, $3 = 1 + 1 + 1$, $4 = 2 + 2$, $6 = 2 + 2 + 1 + 1$ are not distinct.

Note that a partition can be odd and distinct at the same time ($4 = 3 + 1$), only odd ($2 = 1 + 1$), only distinct ($6 = 3 + 2 + 1$), or neither odd nor distinct ($6 = 2 + 2 + 1 + 1$). This observation will be important to what follows.

Our next task was to find all distinct and all odd partitions in the list of parts which we had generated earlier. Here's the extended table

odd	distinct
5	5
3,1,1	4,1
1,1,1,1,1	3,2

odd	distinct
5,1	6
3,3	5,1
3,1,1,1	4,2
1,1,1,1,1,1	3,2,1

odd	distinct
7	7
5,1,1	6,1
3,3,1	5,2
3,1,1,1,1	4,3
1,1,1,1,1,1,1	4,2,1

odd	distinct
7,1	8
5,3	7,1
5,1,1,1	6,2
3,3,1,1	5,3
3,1,1,1,1,1	5,2,1
1,1,1,1,1,1,1,1	4,3,1

odd	distinct
9	9
7,1,1	8,1
5,1,1,1	7,2
5,3,1	6,3
3,3,3	6,2,1
3,3,1,1,1	5,4
3,1,1,1,1,1,1	5,3,1
1,1,1,1,1,1,1,1,1	4,3,2

Table 1. Odd and Distinct partitions for $n = 5, 6, 7, 8, 9$

n	1	2	3	4	5	6	7
p(n)	1	2	3	5	7	11	15
# odd	1	1	2	2	3		
# dist.	1	1	2	2	3		

For example, for $n = 4$ partitions $4 = 1 + 1 + 1 + 1$ and $4 = 3 + 1$ are odd while $4 = 3 + 1$ and $4 = 4 + 0$ are distinct.

Problem: Complete the last two rows of the above table if you have not done so in class.

The pattern in the last two rows of the table was impossible to ignore – *for each integer the number of odd partitions is equal to the number of distinct partitions!* As it turns out this is not a coincidence, but a mathematical theorem, which (together with similar statements about partitions) we shall try to understand in the following weeks.

From the last two rows of the table the pattern is clear – *the number of odd partitions is equal to the number of distinct partitions.* We now can prove this remarkable conjecture. In other words, we need to show that for any integer n for any odd partition there should be only one distinct partition and vice versa – for any distinct partition there is a unique odd partition. Mathematicians say the there is a *one-to-one correspondence* between the set of odd partitions and the set of distinct partitions.

We started with listing odd and distinct partitions for every integer. For small numbers $n = 1, 2, 3, 4$ there are very few partitions of each kind, so we started making tables for numbers 5 and higher (see Table 2).

One needs to come up with a rule which will provide the desired 1-1 correspondence between left and right columns of the above partitions. This rule must be universal – it should work in the same way for any integer and for any partition. In class we started to make some guesses, but did not go far enough.

2.1. **Odd and Distinct.** Next lecture, after understanding the above connection, we shall study partitions which are odd and distinct at the same time. They will be related to some new type of partitions. Below I list all partitions of $n = 9, 10, 11$. The lists are getting quite long, so at this point it is not worth drawing Young diagrams.

Problem: Find all partitions from the lists below which are *simultaneously* odd and distinct:

n=9, p(9)=30

$\{9\}, \{8,1\}, \{7,2\}, \{7,1,1\}, \{6,3\}, \{6,2,1\}, \{6,1,1,1\}, \{5,4\}, \{5,3,1\}, \{5,2,2\}, \{5,2,1,1\},$
$\{5,1,1,1,1\}, \{4,4,1\}, \{4,3,2\}, \{4,3,1,1\}, \{4,2,2,1\}, \{4,2,1,1,1\}, \{4,1,1,1,1,1\}, \{3,3,3\},$
$\{3,3,2,1\}, \{3,3,1,1,1\}, \{3,2,2,2\}, \{3,2,2,1,1\}, \{3,2,1,1,1,1\}, \{3,1,1,1,1,1,1\}, \{2,2,2,2,1\},$
$\{2,2,2,1,1,1\}, \{2,2,1,1,1,1,1\}, \{2,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1\}$

n=10, p(10)=42

$\{\{10\}, \{9,1\}, \{8,2\}, \{8,1,1\}, \{7,3\}, \{7,2,1\}, \{7,1,1,1\}, \{6,4\}, \{6,3,1\}, \{6,2,2\}, \{6,2,1,1\},$
$\{6,1,1,1,1\}, \{5,5\}, \{5,4,1\}, \{5,3,2\}, \{5,3,1,1\}, \{5,2,2,1\}, \{5,2,1,1,1\}, \{5,1,1,1,1,1\},$
$\{4,4,2\}, \{4,4,1,1\}, \{4,3,3\}, \{4,3,2,1\}, \{4,3,1,1,1\}, \{4,2,2,2\}, \{4,2,2,1,1\}, \{4,2,1,1,1,1\},$
$\{4,1,1,1,1,1,1\}, \{3,3,3,1\}, \{3,3,2,2\}, \{3,3,2,1,1\}, \{3,3,1,1,1,1\}, \{3,2,2,2,1\}, \{3,2,2,1,1,1\},$
$\{3,2,1,1,1,1,1\}, \{3,1,1,1,1,1,1,1\}, \{2,2,2,2,2\}, \{2,2,2,2,1,1\}, \{2,2,2,1,1,1,1\},$
$\{2,2,1,1,1,1,1,1\}, \{2,1,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1,1\}\}$

n=11, p(11)=56

$\{\{11\}, \{10,1\}, \{9,2\}, \{9,1,1\}, \{8,3\}, \{8,2,1\}, \{8,1,1,1\}, \{7,4\}, \{7,3,1\}, \{7,2,2\}, \{7,2,1,1\},$
$\{7,1,1,1,1\}, \{6,5\}, \{6,4,1\}, \{6,3,2\}, \{6,3,1,1\}, \{6,2,2,1\}, \{6,2,1,1,1\}, \{6,1,1,1,1,1\}, \{5,5,1\},$
$\{5,4,2\}, \{5,4,1,1\}, \{5,3,3\}, \{5,3,2,1\}, \{5,3,1,1,1\}, \{5,2,2,2\}, \{5,2,2,1,1\}, \{5,2,1,1,1,1\},$
$\{5,1,1,1,1,1,1\}, \{4,4,3\}, \{4,4,2,1\}, \{4,4,1,1,1\}, \{4,3,3,1\}, \{4,3,2,2\}, \{4,3,2,1,1\}, \{4,3,1,1,1,1\},$
$\{4,2,2,2,1\}, \{4,2,2,1,1,1\}, \{4,2,1,1,1,1,1\}, \{4,1,1,1,1,1,1,1\}, \{3,3,3,2\}, \{3,3,3,1,1\}, \{3,3,2,2,1\},$
$\{3,3,2,1,1,1\}, \{3,3,1,1,1,1,1\}, \{3,2,2,2,2\}, \{3,2,2,2,1,1\}, \{3,2,2,1,1,1,1\}, \{3,2,1,1,1,1,1,1\},$
$\{3,1,1,1,1,1,1,1,1\}, \{2,2,2,2,2,1\}, \{2,2,2,2,1,1,1\}, \{2,2,2,1,1,1,1,1\}, \{2,2,1,1,1,1,1,1,1\},$
$\{2,1,1,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1,1,1\}\}$

Problem 2. Odd and distinct parts. Find a rule which identifies odd and distinct partitions from Table 2.

Hint: Start with distinct partitions. Recall that some of them are odd already. So, perhaps, we can just leave them like that – say $6 = 5 + 1$ is odd already. Now let's take a distinct partition which has some even numbers in it, say $8 = 5 + 2 + 1$. In this partitions parts 5 and 1 are already odd, so, again, we can leave them out. What shall we do with 2? What shall we do with 6 in $7 = 6 + 1$, etcetera?

Once you understand how to get an odd partition from a distinct partition go the other way. Which distinct partition does $5 = 3 + 1 + 1$ or $6 = 1 + 1 + 1 + 1 + 1 + 1$ correspond to?

2.2. **Proof of Odd vs. Distinct.** Our first task in class was to prove that for any integer n the number of distinct partitions is equal to the number of odd partitions as the table below suggests:

n	1	2	3	4	5	6	7	8	9
p(n)	1	2	3	5	7	11	15	22	30
# odd	1	1	2	2	3	4	5	6	8
# dist.	1	1	2	2	3	4	5	6	8

We started with listing odd and distinct partitions for every integer. For small numbers $n = 1, 2, 3, 4$ there are very few partitions of each kind, so we started making tables for numbers 5 and higher (see Table 2).

odd	distinct
5	5
3,1,1	4,1
1,1,1,1,1	3,2

odd	distinct
5,1	6
3,3	5,1
3,1,1,1	4,2
1,1,1,1,1,1	3,2,1

odd	distinct
7	7
5,1,1	6,1
3,3,1	5,2
3,1,1,1,1	4,3
1,1,1,1,1,1,1	4,2,1

odd	distinct
7,1	8
5,3	7,1
5,1,1,1	6,2
3,3,1,1	5,3
3,1,1,1,1,1	5,2,1
1,1,1,1,1,1,1,1	4,3,1

odd	distinct
9	9
7,1,1	8,1
5,1,1,1,1	7,2
5,3,1	6,3
3,3,3	6,2,1
3,3,1,1,1	5,4
3,1,1,1,1,1,1	5,3,1
1,1,1,1,1,1,1,1,1	4,3,2

TABLE 2. Odd and Distinct partitions for $n = 5, 6, 7, 8, 9$

One needs to come up with a rule which will provide the desired 1-1 correspondence between left and right columns of the above partitions. This rule must be universal – it should work in the same way for any integer and for any partition.
From Distinct to Odd. First, consider a distinct partition from any of the right columns from the table. If this partition happens to be odd (remember, all numbers are odd), i.e. $5, 3, 1$ or 7, then we do not need to do anything about it – exactly the same partition exists in the left column. All other distinct partitions will have some even numbers in them, like $7, 2$ or $4, 3, 1$. Again we only need to worry about even numbers in these partitions and leave out the odd numbers. After a short discussion in class we agreed on the following rules:

$$\begin{aligned}
2 &\to 1, 1 \\
4 &\to 1, 1, 1, 1 \\
6 &\to 3, 3 \\
8 &\to 1, 1, 1, 1, 1, 1, 1, 1
\end{aligned}$$

Then the students were quickly able to explain the rule – each even number can be divided by 2. If the result is odd, as in $6 : 2 = 3$, then we represent this number as the sum of two odd numbers ($6 = 3 + 3$ or $10 = 5 + 5$). If the result of the division is even, we divide by 2 again. We stop the division process until we get an odd number. We can see that if the initial even number was a power of 2 (2,4,8,16, etc) then we will divide it by 2 until we get 1. This is why $4 = 1 + 1 + 1 + 1$ and 8 is a sum of eight 1s.

Keeping the above rule in mind we have the following matching of the partitions of 8:

$$\begin{aligned}
8 &\to 1, 1, 1, 1, 1, 1, 1, 1 \\
7, 1 &\to 7, 1 \\
6, 2 &\to 3, 3, 1, 1 \\
5, 3 &\to 5, 3 \\
5, 2, 1 &\to 5, 1, 1, 1 \\
4, 3, 1 &\to 3, 1, 1, 1, 1, 1
\end{aligned}$$

Note that we may need to reorder the terms to keep the resulting odd partitions in the standard form (numbers are listed in non-increasing order).
From Odd to Distinct. Now we need to find a map from every partition in the left column (odd) to a partition from the right column (distinct) such that no two partitions are mapped to the same one.

First, if an odd partition is distinct already then we simply map to the same partition in the right column.

Second, if we have some repetitive numbers, like in partition $3, 1, 1, 1, 1$ of 7, we need to combine them in such a way that the resulting partition will be distinct. There is obviously some ambiguity in this process. Indeed, four 1s can be combined either into

$3, 1$ or into 4 (they cannot be combined into $2, 2$ as this is not distinct). In class we kept both options open until we tried to match the partitions explicitly from Table 2. Turns out $1, 1, 1, 1 \to 3, 1$ is not a good option. Indeed, we already have 3 in some partitions, so grouping four 1s into 3 and 1 will not make the resulting partition distinct. Similarly, we should not group six 1s intro 5 and 1 as some odd partitions may have 5 in them already. In short, one should avoid all odd numbers except 1 (the latter is inevitable if we deal with the odd amount of 1s in the odd partition).

After some discussion we proposed the following dictionary of mapping groups of 1s into distinct subpartitions:

$$
\begin{aligned}
1 &\to 1\\
1,1 &\to 2\\
1,1,1 &\to 2,1\\
1,1,1,1 &\to 4\\
1,1,1,1,1 &\to 4,1\\
1,1,1,1,1,1 &\to 4,2\\
1,1,1,1,1,1,1 &\to 4,2,1\\
1,1,1,1,1,1,1,1 &\to 8\\
1,1,1,1,1,1,1,1,1 &\to 8,1
\end{aligned}
$$

This rule can be summarized as follows. Denote the amount of 1s by L. If L is even then we group L 1s into groups of two 1s. For instance, for six 1s we have

$$1,1,1,1,1,1 \to (1,1)\,(1,1)\,(1,1)$$

Then we count the number of these groups. If this number is odd (as it is above) then we leave out one group and combine the others into groups of four 1s

$$(1,1)\,(1,1)\,(1,1) \to (1,1,1,1)\,(1,1) = 4,2$$

We can keep the process of grouping until we have only one group left.

Now assume that L is odd. Then $L-1$ is even and we proceed as above. For instance nine 1s become

$$1,1,1,1,1,1,1,1,1 \to (1,1)\,(1,1)\,(1,1)\,(1,1),\ 1 \to (1,1,1,1)\,(1,1,1,1),\ 1 \to (1,1,1,1,1,1,1,1),\ 1 = 8,1$$

Since most kids knew about powers of two we immediately understood that after applying the above rules the numbers which appear in the resulting distinct partitions are powers of two – $2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8, \ldots$. Therefore we have (re)discovered *binary* presentation of integers – each number (in this problem the number of times 1 appears – the so-called *multiplicity*) is a sum of powers of two:

$$3 = 2^1 + 2^0\,, \qquad 5 = 2^2 + 2^0\,, \qquad 8 = 2^3\,.$$

Finally we need to address other repetitive odd integers from the left columns. If we have, say, $5, 5, 5$ or $7, 7, 7, 7$ inside an odd partition we can first replace them with 1s and proceed as above and then multiply the resulting distinct partitions by the corresponding odd number. For example,

$$5+5+5 = 5 \times (1+1+1) \to 5 \times (2+1) = 10+5$$

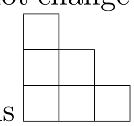
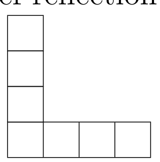
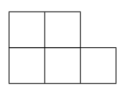
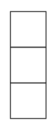
or

$$7+7+7+7 = 7 \times (1+1+1+1) \to 7 \times 4 = 28$$

This concludes the proof.

2.3. **Odd & Distinct.** Now let us study partitions which are odd and distinct at the same time. They will be related to some new type of partitions. Previously I listed all partitions of $n = 9, 10, 11$ and more partition are given in the accompanying PDF file. We can see that imposing both odd and distinct conditions at the same time leaves out only very few partitions. Thus for $n = 9$ we only have $\{9\}$ and $\{5,3,1\}$, for $n = 10$ we get $\{9,1\}$ and $\{7,3\}$, while for $n = 11$ one gets $\{11\}$ and $\{7,3,1\}$.

3. Symmetric Partitions

Towards the end of the class we talked about symmetric partitions which do not change under reflection along the diagonal (see the Keynote file). In particular, partitions  and  are symmetric, but partitions  or  are not.

In class we counted symmetric partitions for some integers and found that their number coincides with the number of odd and distinct partitions for the same integer! Based on our experience this cannot be a coincidence and it should hold in general.

Problem 3. Odd and distinct vs. Symmetric. Explain why for each integer n the number of odd and distinct partitions of n is equal to the number of symmetric partitions of n.

Hint: Draw Young diagrams for each type of partitions. You may use the list of partitions from the accompanying PDF for that.

Our new task was to study integer partitions which are odd and distinct at the same time. We observed previously that they should match with symmetric partitions. Now we are ready to prove that this is indeed the case.

The students were puzzled in the beginning so we started to list all symmetric and all odd and distinct partitions on the board. We skipped small integers and started with $n = 7$, see Table 3. We used toy blocks as well as grid paper to visualize partitions in the left column and lists of integer partitions from the last homework to understand the right column.

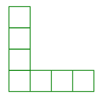
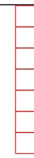
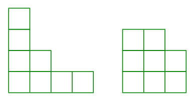
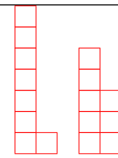
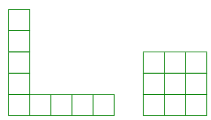
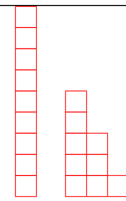
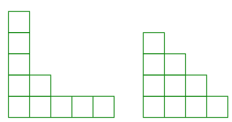
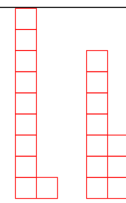
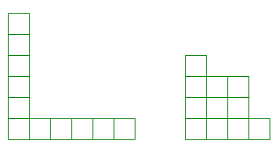
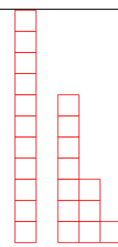
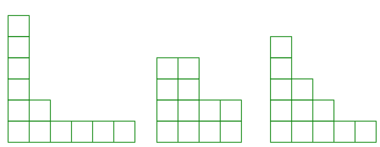
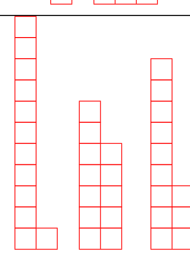
n	Symmetric	Odd&Distinct
7		
8		
9		
10		
11		
12		

TABLE 3. List of symmetric vs. odd and distinct partitions for $n = 7, \ldots, 12$.

We can see that the number of partitions in both columns of Table 3 grows much slower than the number of partitions with only odd parts:

n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Then we tried to understand how using these pictures to match the two columns. We did not finish so this task carries over to the homework.

First, we need to finish the proof of the above statement.

Problem 4. Odd and distinct vs. Symmetric. Explain why for each integer n the number of odd and distinct partitions of n is equal to the number of symmetric partitions of n.

You can easily check that you understand how to prove it if you can easily find which symmetric partition corresponds to a given odd and distinct partition and vice versa.

Say, you are given partition $\{5, 4, 3, 2, 1\}$ of 15 which is symmetric (if in doubt draw its Young diagram, it looks like a staircase). Which odd and distinct Young diagram (or partition) will it correspond to?

Conversely, say we have $\{11, 5, 3, 1\}$ which is odd and distinct. Find the corresponding symmetric partition.

3.1. **Summary table.** We started our class by reviewing our progress up to date which is summarized in the table below

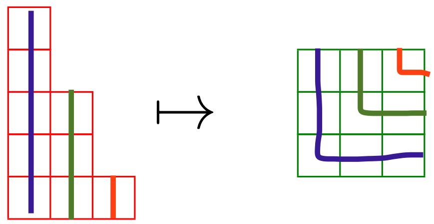
n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Thus the number of odd partitions is equal to the number of distinct partitions and the number of symmetric matches the number of odd and distinct.

3.2. **Proof that Odd&Distinct = Symmetric.** In order to finish the proof from the last class we nee to construct the map backwards – given any odd and distinct partition we should be able to construct a unique symmetric partition. Keep in mind that the one-to-one correspondence exists provided that we can make identifications in both directions!

In this case the inverse mapping is more or less straightforward – we can merely reverse the arrows in the previous argument. Indeed, since each part of the red partition is odd we can 'bend' each column into an L-shaped hook. Since all parts are distinct we can then stack all hooks one on top of each other in the same order as in the partition.

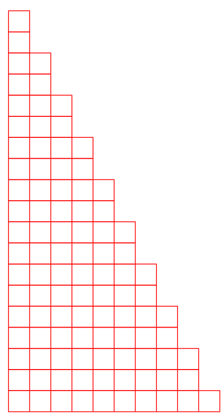
For instance in the example below we have partition $\{5,3,1\}$ becomes symmetric partition $\{3,3,3\}$. The number of parts in the red partition is equal to the number of hooks in the green partition:



$\blacksquare$

If you understood the proof solve the following

Problem: Find symmetric partition which corresponds to the following odd and distinct partition $\{19,17,15,13,11,9,7,5,3,1\}$



3.3. **First part $>$ Second part.** Our next problem was the following: *Count partitions of n such that the first part is strictly greater than the second.* For instance, for partitions $\{3,2,1\},\{5,2,1,1\},\{4\}$ this condition is satisfied (if there is only one part, as in $\{4\}$ we count it as well). However, for partitions $\{2,2,1\},\{3,3,3\}$ the first and second parts are equal, so we do not count them. Notice that we do not worry about how second and third, third and fourth, etc parts are related.

We used the first slide of the Keynote to do the counting and quickly came up with the following table (here we denoted the total number of such partitions with $Q(n)$)

n	1	2	3	4	5	6	7	8
Q(n)	1	1	2	3	5	7	11	15

In particular, for $n=4$ out of five partitions only in $\{2,1,1\},\{3,1\}$ and $\{4\}$ their first parts are strictly greater than the second parts. We then asked the students if they have seen these numbers earlier and the answer was affirmative – these are the numbers of *all* partitions of n albeit shifted by one position. Indeed, if we complete the above table we get

n	1	2	3	4	5	6	7	8
Q(n)	1	1	2	3	5	7	11	15
p(n)	1	2	3	5	7	11	15	22

Or, in other words, $Q(n) = p(n-1)$. We then needed to explain *why* this happens.

Problem: Show that this is indeed the case.

Problem 5. Consider the following two types of partitions of n:

(1) Partitions whose parts are *not divisible by 3.* For instance:
$\{2, 2, 2, 2\}, \{5, 1, 1, 1\}, \{2, 2, 1, 1, 1, 1\}$

(2) Partitions in which each part is *not repeated 3 or more times.* For example:
$\{5, 3\}, \{3, 3, 1, 1\}, \{3, 2, 2, 1\}$

Find out if there is any connection between (1) and (2).

Problem 6. Find the sum $2 + 4 + 6 + ... + 198 + 200$.

Problem 7. How many consecutive odd numbers are in the sum $1+3+5+\cdots+117+119$?
Find the following sum of consecutive odd numbers $1 + 3 + 5 + \cdots + 117 + 119$.

4. Binaries

We started our class with reminding ourselves about binary presentation of integers. In fact, we already used the results of the following problem two classes ago when we compared odd partitions with distinct partitions.

Problem 8. Powers of two. Consider the list of numbers $1, 2, 4, 8, 16, 32, 64, 128, \dots$. Recall that in class we understood that these are integer powers of two, i.e. $4 = 2^2, 32 = 2^5$ and by convention $1 = 2^0$ (the only odd number in the list).

Can you figure out a way to add up some of these numbers taking each number at most one time to get 93, 62, 127?

Is there more than one way to do that?

The students quickly found

$$93 = 64 + 16 + 8 + 4 + 1,$$
$$62 = 32 + 16 + 8 + 4 + 2,$$
$$127 = 64 + 32 + 16 + 8 + 4 + 2 + 1.$$

Most of the students intuitively knew that the above solution is unique, however, it took some time to say it out loud. The key was that if you look at the sums of the first one, two, three, four, etc. of the powers of two, we get the following:

$1 = 1 < 2,$
$1 + 2 = 3 < 4,$
$1 + 2 + 4 = 7 < 8,$
$1 + 2 + 4 + 8 = 15 < 16,$
$1 + 2 + 4 + 8 + 16 = 31 < 32,$
$1 + 2 + 4 + 8 + 16 + 32 = 63 < 64,$
$1 + 2 + 4 + 8 + 16 + 32 + 64 = 127 < 128,$

So, for example, when we write $93 = 64 + 16 + 8 + 4 + 1$, if we try to replace the 64 with some of the other numbers, we cannot do it because even using all the numbers less than 64 in the list, we cannot get all the way up to 64, and by then we will have used up all eligible numbers! The same can be said for 16, 8, 4, and 1. So any number that can be written as a sum of these, can be written as a sum of these in exactly one way!

Problem 9. Calculate the sum $1 + 2 + 3 + ... + 1000$.

Problem 10. Calculate $1 + 5 + 9 + 13 + 17 + 21 + 25$?
$1 + 5 + 9 + 13 + \cdots + 401$? Note that in both sums we pick every other odd number. Hint: Use the connection between odd&distinct and symmetric partitions.

Problem 11. Divisibility by 4 and 5. Show that the Glashier's theorem (see previous lecture notes, there we proved it for parts which are not divisible by 3) holds for $n = 4$ or $n = 5$. You may use partitions of n from the list that we gave you couple of lectures ago.

5. Sums from Partitions. Figurate Numbers

Last lecture have proven the formula using square numbers:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100\,, \tag{1}$$

so that the sum of the first ten odd integers is equal to $10^2 = 10 \times 10 = 100$. Or, more generally, if we sum all k odd numbers which appear between 1 and $2k$ we get

$$1 + 3 + \cdots + (2k - 1) = k^2\,.$$

5.1. **Gauss Method.** There is also another way of counting sums like this using the *Gauss*' method which was reportedly invented by early Karl F. Gauss (German mathematician) when he was in elementary school. Let us look at equation (1) to illustrate the method (Gauss himself summed $1 + 2 + 3 + \cdots + 99 + 100$, see below). There are ten numbers in the sum which can be combined into five paris

$$1 + 19, \qquad 3 + 17, \qquad 5 + 15, \qquad 7 + 13, \qquad 9 + 11,$$

Clearly the sum of the numbers in each pair is 20. Since there are five pairs the answer is 100.

Consider another example: $1 + 2 + 3 + \cdots + 99 + 100$

Again, we can combine first and the last number – $1 + 100$, second and the one before last $2 + 99$, etcetera. Each pair amounts to 101 and there are 50 pair in total hence the answer is $50 \times 101 = 5050$.

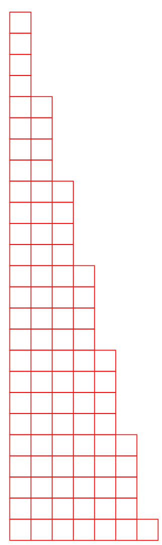
Here in both examples we had even amount of numbers in the sums. If the total amount is odd, there will be one number remaining in the middle, which should be added in the end, see below.

5.2. **Triangular Numbers.** Now let us calculate

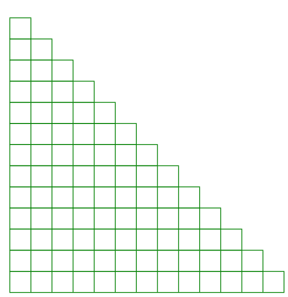
$$1 + 5 + 9 + 13 + 17 + 21 + 25$$

where we sum every other odd number between 1 and 25 (skipping 3,7,11,15,19 and 23). The students quickly calculated the sum and obtained 91. If we use Gauss method we'll have three pairs $1 + 25, 5 + 21$ and $9 + 17$ equal to 26 each and we need to add 13 in from the middle to get $3 \times 26 + 13 = 91$.

It is instructive to do the same exercise using partitions. Consider odd and distinct partition {25,21,17,13,9,5,1}.

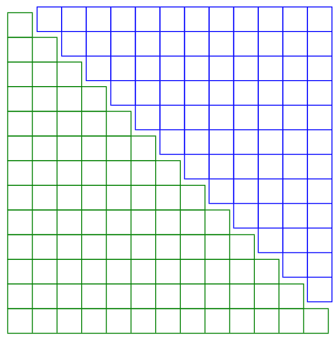


If we convert it to a symmetric partition (check Lecture 4 if you forgot) we'll get the following partition {13,12,11,10,9,8,7,6,5,4,3,2,1}



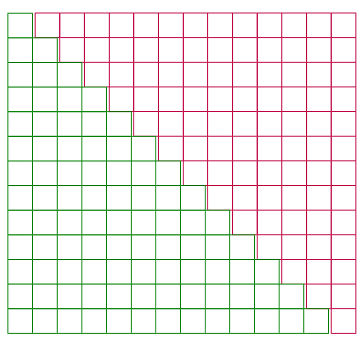
So the Young diagram has a shape of a triangle (well, almost, it's more of a staircase, however, if you put dots in the middle of blocks then it will become an actual right triangle), hence the name – triangular number.

How can we know the number of blocks in this triangle without counting? Students quickly realized that we need to complete this shape to a square or to a rectangle. In the first case we can put on top of the above green diagram another triangular diagram which is smaller by one row and one column



This way we get a square 13×13 which has 169 blocks in it. However, we only need to count green blocks. Clearly, one cannot divide 169 by 2 to get an integer. Therefore, we need to remove the diagonal which contains 13 green blocks so the square without the diagonal has 169-13= 156 blocks. Half of that is 78 (cf. with the calculation above using Gauss' method). Finally we add the diagonal to complete the answer: $78+13 = 91$.

Yet a faster way to solve the problem is to complete the staircase to a rectangle instead simply by sliding a mirror copy of the same staircase from above:



In this figure the rectangle has width 14 and height 13, it contains 13×14 blocks. However, we only need to count green blocks which is exactly one half: $\frac{13\times 14}{2} = 13 \times 7 = 91$.

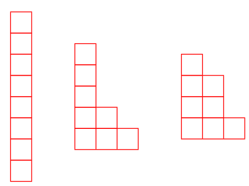
5.3. **More fun with partitions.** Our next story is about how many parts each partition of an integer has. Say, partition $\{7, 6, 5, 1\}$ has four parts and four is an even number. Or $\{9, 8\}$ has two parts, and two is even. Thus we can put them into a type of partitions with *even number of parts*. Analogously we can count partitions with *odd number of parts*, i.e. $\{8\}$ has one part, $\{9, 7, 2\}$ has three parts or $\{5, 4, 3, 2, 1\}$ has five parts – all numbers of parts are odd.

Problem: Count and draw partitions with *odd distinct parts* and with *even distinct parts* for $n = 5, 6, 7, 8, 9, 10, 11, 12, 13$. You may use lists of partitions provided to you in earlier lectures. Explain the pattern that you see.

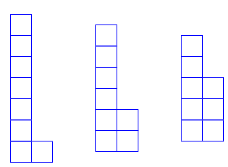
In class we only started drawing their Young diagrams on the board and found that for *some* numbers the number of partitions of both types is the same, however, for some integers the number is different by one.

In the separate attachment you can find the list of partitions for $n = 7, 8, 9$. Notice that for $n = 8$ and $n = 9$ both types contain equal number of partitions, however, for $n = 7$ there are two odd and three even partitions.

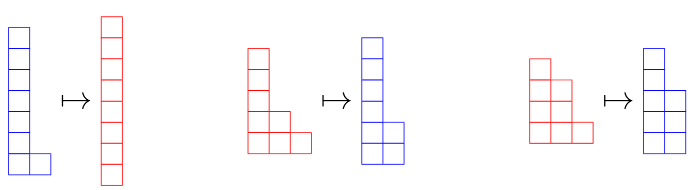
In class we drew Young diagrams to illustrate both types of partitions and tried to find a match between the two types where possible (we still need to explain the mismatch for $n = 7$ and $n = 12$ which we also did in class). For instance, for $n = 8$ we have the following distinct odd partitions (again, here odd means with odd numbers of parts, we don't worry about parity of each individual part)



and the following even-parted partitions



The first list of partitions has odd number of columns (1 or 3), while the second list has two columns. In this case one can easily match three red partitions with three blue partitions by throwing the bottom-right block in the last column (if it is there) on top of the first column:



Try this trick with distinct partitions for other n (see the attachment for higher n). See why $n = 5, 7, 12, \ldots$ are special.

Problem 12. Calculate the sum $1 + 2 + 3 + \cdots + 999$.

Problem 13. Calculate the sum $3 + 6 + 9 + 12 + 15 \cdots + 75$.

Problem 14. Calculate
$1 + 5 + 9 + 13 + \cdots + 401$
Note that in this sum we pick every other odd number.

5.4. **Triangular and Square Numbers.** In class we have discussed triangular, square and pentagonal numbers in connection with integer partitions. While the appearance of the former two numbers is not surprising since a symmetric partition can take a shape of a triangle or a square (see previous lectures and the keynote file for figures) it is not clear yet how pentagons may arise.

We started with filling in the table of triangular T_n and square S_n numbers and tried to spot patterns in what we saw

n	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	1	3	6	10	15	21	28	36	45	55	66	78	91
S_n	1	4	9	16	25	36	49	64	81	100	121	144	169

Here T_n stands for the n-th triangular number, so $T_5 = 15, T_9 = 45$, etcetera. Same for square numbers: $S_4 = 16, S_12 = 144$. You first locate the value of n in the table and then find the corresponding values of T_n and S_n below it.

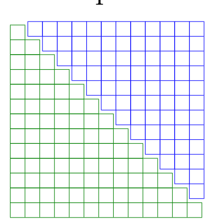
The students quickly found the following patterns

Sum of two neighboring triangular numbers is a square number.

$$T_n + T_{n+1} = S_{n+1}\,, \tag{2}$$

or that the sum of the two neighboring triangular numbers equals pentagon number, i.e. $T_3 + T_4 = S_4$ which is $6 + 10 = 16 = 4 \times 4$; or $45 + 55 = 100 = 10 \times 10$.

We recalled that we even know the proof of the this result from the previous lecture. The proof can be illustrated by the following picture – two staircase-shaped triangles can be fit together as a jig-saw puzzle intro a square:



In this example we have $T_{12} + T_{13} = S_{13}$ or $78 + 91 = 169$.

Recursive formula for triangular numbers. From the picture of triangles from the slides it is clear that

$$T_n + (n+1) = T_{n+1}\,, \tag{3}$$

for instance, $T_4 + 5 = T_6$, $T_7 + 8 = T_8$, etcetera.

We now ready to write the formula for any triangular number

$$T_n = \frac{n \times (n+1)}{2}\,.$$

Problem: Prove the above formula by induction.

6. Pentagonal Numbers

Then it was time to draw some pentagons. The *pentagonal* or, simply *pentagon*, numbers P_n are numbers which are equal to the number of dots which we can distributed inside a pentagon (see slides). As the corresponding keynote slide explains, the idea is to put dots into vertices of a regular pentagon (that gives $P_2 = 5$, we also assign $P_1 = 1$ for a single dot). On the next step we draw a larger pentagon which contains the previous one and we add dots on the outer edges such that each edge has 3 dots. In other words, the index in P_n tells how many dots are on the edges of the pentagon. Same applied for T_n and S_n.

It took some imagination from kids to carefully draw pentagonal shapes on the paper. After several minutes we added the third row to the table:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	1	3	6	10	15	21	28	36	45	55	66	78	91
S_n	1	4	9	16	25	36	49	64	81	100	121	144	169
P_n	1	5	12	22	35	51	70	92	117	145	176	210	247

Same question – do you see any patterns? Once the table was in front of the students' eyes they spotted the following two rules

$$T_n + S_{n+1} = P_{n+1}\,, \tag{4}$$

for instance, $36 + 81 = 117$ ($T_8 + S_9 = P_9$) or $15 + 36 = 51$ ($T_5 + S_6 = P_6$), as well as

$$T_n + S_n - n = P_n\,, \tag{5}$$

like $10 + 16 - 4 = 22$ ($T_4 + S_4 - 4 = P_4$) or $55 + 100 - 10 = 145$ ($T_{10} + S_{10} - 10 = P_{10}$).

Problem: Can you show that the last equation (7) follows from previous equations (2), (3) and (7)? If you're still hesitant with indices try to look at the table and see what each equation mean.

6.1. Extending to Negative n.

One can formally define triangular, square and pentagonal numbers for negative n using the equations (2), (3), (7), and (7) which we just discussed. Indeed, (3) says that

$$T_1 + 2 = T_2\,, \qquad T_0 + 1 = T_1\,, \qquad T_{-1} + (-1+1) = T_0\,, \qquad T_{-2} + (-2+1) = T_{-1}\,, \ldots$$

Which lead to

$$1 + 2 = 3\,, \qquad T_0 + 1 = 1\,, \qquad T_{-1} + 0 = T_0\,, \qquad T_{-2} + -1 = T_{-1}\,, \ldots$$

which leads to $T_0 = T_{-1} = 0$ and $T_{-2} = 1$.

Problem: Using this idea find T_n, S_n and P_n for some negative n. Use all four above equations to check your results.

Problem 15. Pentagonal partitions. Using partitions of integers in handouts and what we have discussed in class so far find and draw other pentagonal partitions for $n = 1, 2, 5, 7, 12, 15, 22$ and other pentagonal numbers.

In class we have drew the following summary table of triangular T_n, square S_n and pentagonal P_n numbers including negative indices n

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
T_n	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
S_n	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64
P_n	100	77	57	40	26	15	7	2	0	1	5	12	22	35	51	70	92

Here are the patterns:

$$T_n + S_{n+1} = P_{n+1}\,, \tag{6}$$

for instance, $36 + 81 = 117$ ($T_8 + S_9 = P_9$) or $15 + 36 = 51$ ($T_5 + S_6 = P_6$), as well as

$$T_n + S_n - n = P_n\,, \tag{7}$$

like $10 + 16 - 4 = 22$ ($T_4 + S_4 - 4 = P_4$) or $55 + 100 - 10 = 145$ ($T_{10} + S_{10} - 10 = P_{10}$).

Extension to negative n can be done using the above equations

$$T_1 + 2 = T_2\,, \qquad T_0 + 1 = T_1\,, \qquad T_{-1} + (-1 + 1) = T_0\,, \qquad T_{-2} + (-2 + 1) = T_{-1}\,, \ldots$$

Which lead to

$$1 + 2 = 3\,, \qquad T_0 + 1 = 1\,, \qquad T_{-1} + 0 = T_0\,, \qquad T_{-2} + -1 = T_{-1}\,, \ldots$$

which leads to $T_0 = T_{-1} = 0$ and $T_{-2} = 1$. Then from (7) we find that $P_{-1} = 2$, etc.

Now we can write all pentagonal numbers P_n for positive and negative values of n in the increasing order

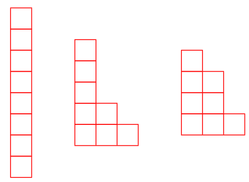
$$(1, 2), \quad (5, 7), \quad (12, 15), \quad (22, 26), \quad (35, 40), \quad (51, 57), \quad (70, 77), \quad (92, 100) \tag{8}$$

Notice that we grouped them in pairs (1,2), (5,7) etc. This will become important later.
Problem: Do you see a pattern in (8)? Why did we break them into pairs? Find the next three pairs.

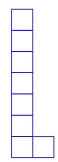
6.2. **Pentagonal Partitions.** One of our goals is to understand the recurrent formula for integer partitions which would allow to calculate $p(n)$ for any n provided that we know $p(k)$ for $k < n$. Some kids are narrowing down on the correct answer (after all, at this stage of our progress it would not hurt to look it up on the internet, i.e. https://en.wikipedia.org/wiki/Pentagonal_number_theorem), yet, we need to develop more technology in order to be able to fully appreciate the beauty of the result.

Fore convenience I repeat the partitions with *odd distinct parts* and with *even distinct parts*. Recall that we had a match almost always, however, for certain n we had one more partition of one kind or another.

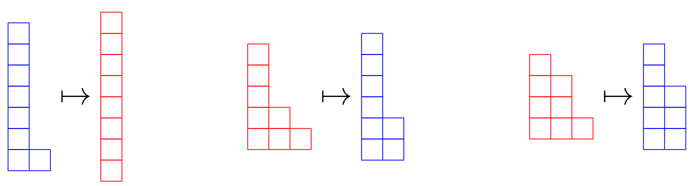
n = 8 : The following 3 partitions have odd number of distinct parts



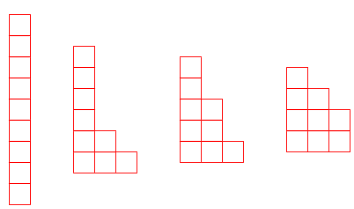
and the following 3 partitions have even number of distinct partitions



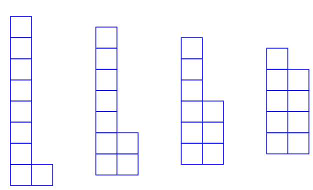
The first list of partitions has odd number of columns (1 or 3), while the second list has two columns. One can easily match three red partitions with three blue partitions by throwing the bottom-right block in the last column (if it is there) on top of the first column:



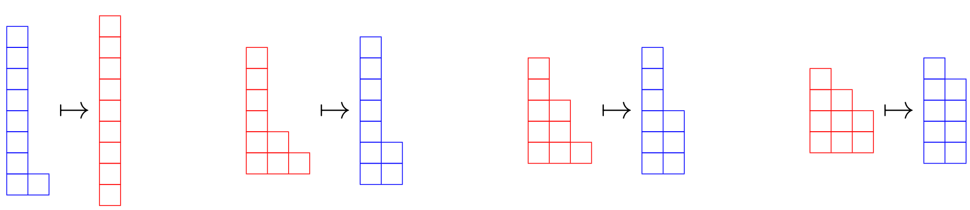
n = 9 : The following 4 partitions have odd number of distinct parts



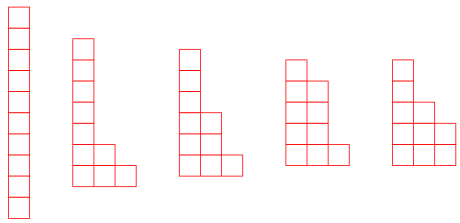
and the following 4 partitions have even number of distinct partitions



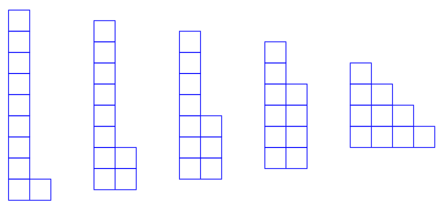
The identification goes as follows



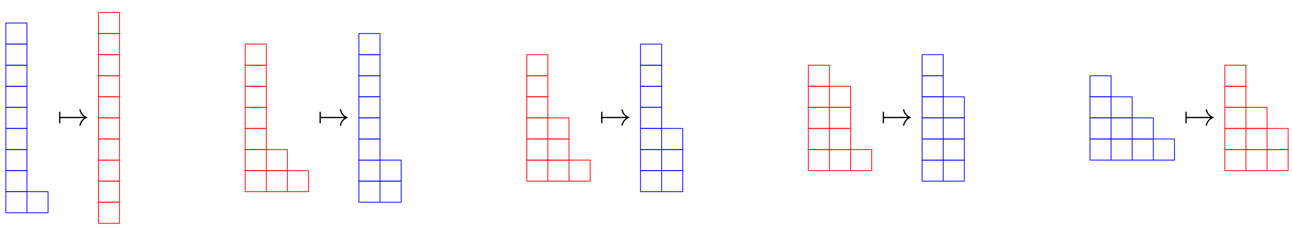
$\mathbf{n = 10}$: The following 5 partitions have odd number of distinct parts



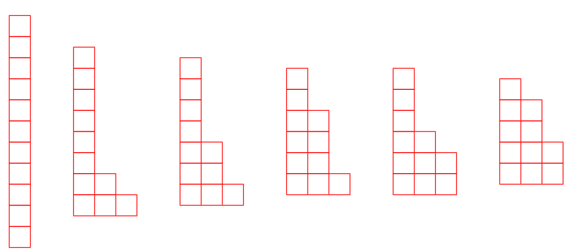
while the following 5 even-parted partitions



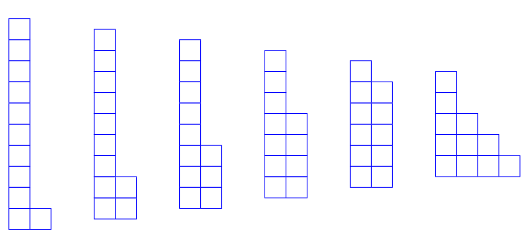
The identification goes as follows



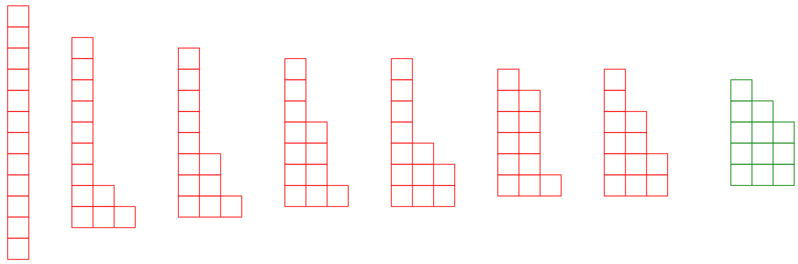
$\mathbf{n = 11}$: The following 6 partitions have odd number of distinct parts



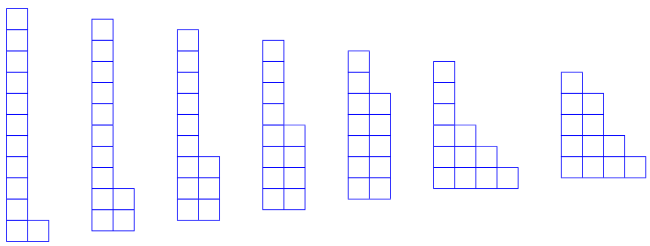
while the following 6 have even number of distinct parts

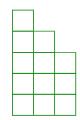


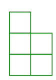
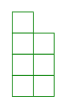
n = 12 : The following 8 partitions have odd number of distinct parts



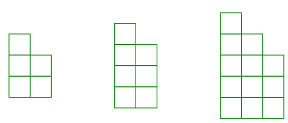
However, we only have 7 partitions with even number of distinct parts



We can see that the following 'pentagon-shaped' diagram  is extra and does not fit the matching pattern.

Several lectures ago we also saw that  for $n = 5$ and  for $n = 7$ were also exceptional and did not fit the matching pattern.

Notice that each of the above green diagrams



is a Young diagram for a pentagonal number.

Problem 16. Pentagonal partitions. Look at formula (8) again. Draw pentagonal shaped partitions (similar to the last green partition above) for all numbers from formula (8). Why are the partitions inside each pair similar to each other?

7. Restricted Partitions

Next we would like to find recurrent relations within partitions with fixed number of parts.

Problem 17. Restricted Partitions. Let us now look at integer partitions of n which have *exactly* 4 parts. From the list of partitions of 7

$$\{\{7\},\{6,1\},\{5,2\},\{5,1,1\},\{4,3\},\{4,2,1\},\{4,1,1,1\},\{3,3,1\},\{3,2,2\},$$
$$\{3,2,1,1\},\{3,1,1,1,1\},\{2,2,2,1\},\{2,2,1,1,1\},\{2,1,1,1,1,1\},\{1,1,1,1,1,1,1\}\}$$

Only these qualify

$$\{\{4,1,1,1\},\{3,2,1,1\},\{2,2,2,1\}\}$$

Make lists for $n = 4, 5, 6$ with all possible restricted parts, i.e. all partitions of 4 with 1, with 2, with 3 parts, etc., same for $n = 5$ and $n = 6$. Count each number of partitions, call it $p_k(n)$. Do you see any pattern?

In class we looked at restricted partitions from $n = 1$ to $n = 10$ and drew table 7

This table is larger than the one from the previous class so we had more information and could observe more patterns. On the video you can see how those new patterns appeared in front of our eyes.

Eventually we have found that if you take a number, say $p_5(10) = 7$, then move to its upper-left corner where $p_4(9) = 6$ resides. Then we need to compensate for missing $7-6 = 1$. There are a lot of 1s in the table, so it may be confusing which 1 to pick. Since we?re after a pattern which should *always* work we need a simple rule which would tell us how to pick the missing number. One suggestion in class was to take $p_5(6) = 1$ works perfectly. Still, there's another 1 on top of it – $p_5(5)$, so we could as well picked that one.

In order to remove the ambiguity let us consider another example. Start with $p_4(10) = 9$, then go towards its top-left corner, find it to be $p_3(9) = 7$. We are short by $9 - 7 = 2$. Luckily for us there's only one 2 corresponding to $p_4(6)$. So we conclude that

$$p_4(10) = p_3(9) + p_4(6).$$

We did couple more examples and convinced ourselves that the rule indeed works, say

$$p_3(11) = p_2(10) + p_3(8) \qquad (10 = 5 + 5)$$

Let?s now try to write this rule for all n-integer and k-number of parts. It states

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k) \tag{9}$$

Here we assume that k is less or equal n, otherwise we'll get a negative number for $n-k$. We concluded that in this case $p_k(n-k)$ should be zero. Indeed, this is correct and it implies that

$$p_k(n) = p_{k-1}(n-1)$$

for $k > n$ One can check this by looking at the right half of the table. That?s why we have diagonals of 1s, 2s, 3s on the right.

	1	2	3	4	5	6	7	8	9	10	11	12
p(1)	1											
p(2)	1	1										
p(3)	1	1	1									
p(4)	1	2	1	1								
p(5)	1	2	2	1	1							
p(6)	1	3	3	2	1	1						
p(7)	1	3	4	3	2	1	1					
p(8)	1	4	5	5	3	2	1	1				
p(9)	1	4	7	6	5	3	2	1	1			
p(10)	1	5	8	9	7	5	3	2	1	1		
p(11)												
p(12)												

FIGURE 1. Number of restricted partitions $p_k(n)$ for $n = 1, \dots, 10$

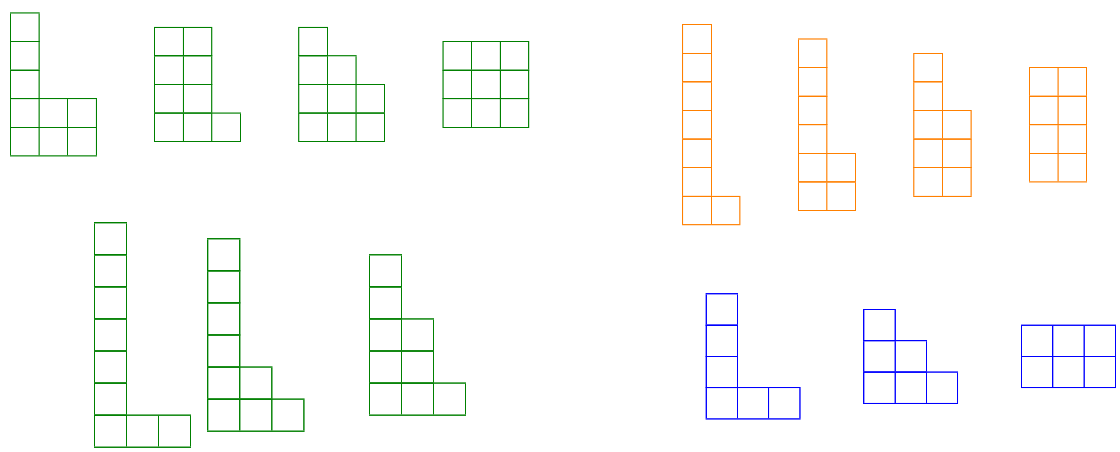
Question 1: Explain that the pattern is correct. Consider an example, say $p_3(9) = p_2(8) + p_3(6)$. Draw the corresponding Young diagrams for the left hand side of the equation and for its right hand side. Try to see how we can move/add/remove blocks to these diagrams to show the equivalence.

In the figure below we drew all the diagrams

Recurrent Formula $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

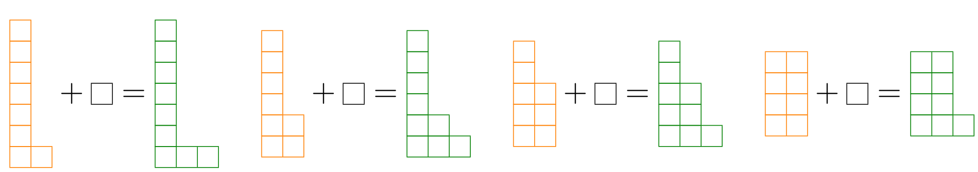
$$p_3(9) = p_2(8) + p_3(6)$$

$$7 \qquad 4 \qquad 3$$

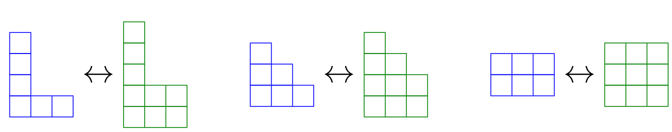


We need to match each of 7 green diagrams on the left side of the picture with a unique diagram of the right side and vice-versa (cf. odd vs. distinct or odd&distinct vs. symmetric earlier in the course).

Notice that $p_3(9)$ counts diagrams with one less block and one less part than $p_2(8)$. What does it mean exactly? We need to add *one* single block to any of the four orange diagrams from $p_2(8)$ so that the new diagrams will have 9 blocks in them and 3 parts (columns). How shall we do that? The only solutions is to add this block on the bottom-right of each orange diagram! Indeed,



Now we need to math the remaining 3 green diagrams of partitions of 9 with 3 blue diagrams of partitions of 6. This goes as follows



As it is clearly seen from the picture we merely need to add one row of three blocks to each of the blue diagrams (equivalently, remove this row from green diagrams). The last operation explains the presence of $p_3(6)$ (or, in general, $p_k(n-k)$) term in the right hand side of the formula. Both types of diagrams in the latter picture have 3 (or k in general) columns and green diagrams' bottom rows are the same. Note that there might not be any diagrams of that kind.

Question 2: Use the pattern which we have found (9) to find all numbers in the next rows in table 7. What are $p_k(11), p_k(12), p_k(13)$ for all k? Once you have found each new $p_k(n)$ calculate the sum. What should it be equal to?

$$p_1(n) + p_2(n) + \cdots + p_n(n)$$

8. Counting Partitions

Having done a lot of preparation, we studied the recurrent formula for integer partitions. At this moment we still don't have a full honest proof of the formula (that would require a little bit of algebra), however, we can explain the result intuitively using pentagonal numbers.

In this note I nevertheless write the full derivation of the Euler's formula which you and your children can study. You are welcome to send follow-up questions over the email if your kids want to understand it in more details. I do realize that most of the material below (in fact, all of it after *1.1 Generating Function for Pentagonal Numbers*) is beyond the scope of grades 1-4, however, I hope that the students got some understanding of where the formula comes from and that it will motivate them to study the subject further. This intuition is built on previous problems which we had solved earlier.

The summary of our last lecture is in the first three pages of this notes.

Problem 18. Recursive formula for the number of partitions. Below you can find the list of total number of integer partitions for $n = 1, \dots, 30$. Using your imagination and what we have learned about Fibonacci numbers last quarter (if you were there) try to find a pattern (recursive formula) in these numbers. Keep in mind that the desired formula is more complicated than the formula for Fibonacci numbers, albeit the idea behind it is similar – express next numbers in the sequence using (some of) previous numbers. For completeness let us denote $p(0) = 1$. Pay special attention to pentagonal numbers.

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627,$$
$$792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, \dots$$

For instance

$$
\begin{aligned}
5 &= 3 + 2\,,\\
11 &= 7 + 5 - 1\,,\\
15 &= 11 + 7 - 2 - 1\,,\\
56 &= 42 + 30 - 11 - 5\,,\\
77 &= 56 + 42 - 15 - 7 + 1\,,\\
101 &= 77 + 56 - 22 - 11 + 1\,.
\end{aligned}
$$

After several lectures of preparation we are ready to understand the recurrent formula for the number of partitions which reads

$$
p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-p(n-22)-p(n-26)+\dots \tag{10}
$$

In class we checked that the formula on several examples (also see the video). We assume that $p(0) = 1$ and $p(m) = 0$ for any negative m. The formula can also be illustrated using the following 'magic ruler':

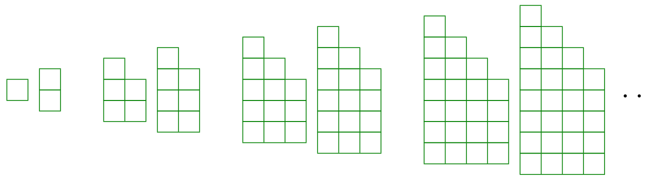
$\rightarrow$	**56**
$+$	42 = $\mathbf{p}(10)$
$+$	30 = $\mathbf{p}(9)$
	22 = $\mathbf{p}(8)$
	15 = $\mathbf{p}(7)$
$-$	11 = $\mathbf{p}(6)$
	7 = $\mathbf{p}(5)$
$-$	5 = $\mathbf{p}(4)$
	3 = $\mathbf{p}(3)$
	2 = $\mathbf{p}(2)$
	1 = $\mathbf{p}(1)$
	1 = $\mathbf{p}(0)$

$\rightarrow$	**77**
$+$	56 = $\mathbf{p}(11)$
$+$	42 = $\mathbf{p}(10)$
	30 = $\mathbf{p}(9)$
	22 = $\mathbf{p}(8)$
$-$	15 = $\mathbf{p}(7)$
	11 = $\mathbf{p}(6)$
$-$	7 = $\mathbf{p}(5)$
	5 = $\mathbf{p}(4)$
	3 = $\mathbf{p}(3)$
	2 = $\mathbf{p}(2)$
	1 = $\mathbf{p}(1)$
$+$	1 = $\mathbf{p}(0)$

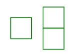
Pluses and minuses in the ruler on the left are placed at locations of pentagonal numbers.

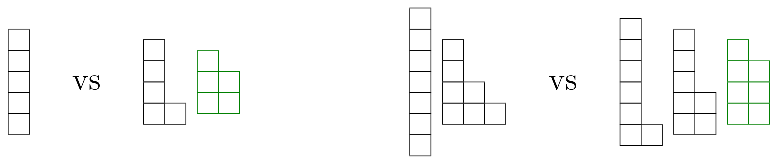
The numbers in red are by now familiar pentagonal numbers which have the following partitions

$$
(P_1, P_{-1}) \quad (P_2, P_{-2}) \quad (P_3, P_{-3}) \quad (P_4, P_{-4}) \ \dots
$$

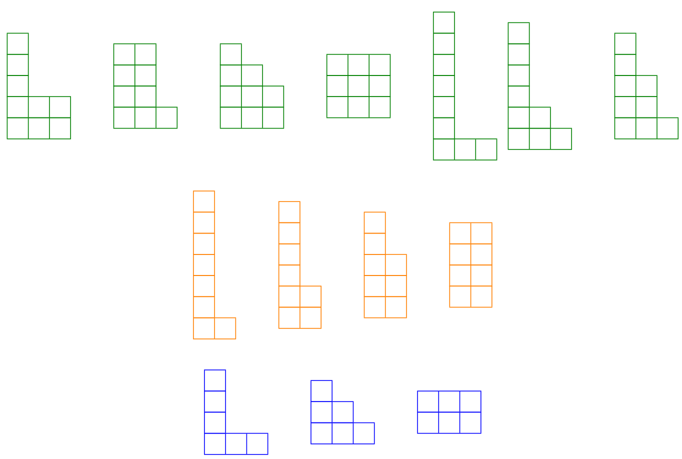


Pentagonal numbers are special for many reasons, most importantly for us they mark integers for which the number of partitions with even distinct parts is not equal to the number of partitions with odd distinct parts.

Indeed, for $n = 1$ and $n = 2$ we have one partition  with one (odd) part and none with two parts. For $n = 5$ and $n = 7$ we have by one more partition with even number of distinct parts than that with odd number:



For $n = 12$ and $n = 15$ it is the other way around – partitions with odd number of distinct parts exceed partitions with even number of distinct parts by one (see previous lecture for diagrams), etcetera. Thus the parity pattern alternates every other pentagonal number. This is the reason for two pluses, two minuses, two pluses, etc. in Euler formula (10).



Therefore the following statement holds. We demonstrated its validity earlier by matching odd and even partitions by moving blocks from top to bottom and back.

Theorem: If n is not a pentagonal number, then the number of even distinct partitions of n, call it $q_e(n)$ equals the number of odd distinct partitions of n, call it $q_o(n)$. So $q_e(n) = q_o(n)$ and so the total number of distinct partitions of n, call it $q(n)$ is $q(n) = 2q_o(n)$ which is even.

If n is a pentagonal number, say $n = P_j$, then $q_e(n) = q_o(n) + (-1)^j$ and so $q(n) = 2q_o(n) + (-1)^j$ which is odd.

Problem: Show that $P_n = \frac{n(3n-1)}{2}$. You may use the fact that a pentagonal number is a sum of a square number $S_n = n^2$ and a triangular number $T_{n-1} = \frac{n(n-1)}{2}$ or use induction.

8.1. **Generating Function for Pentagonal Numbers.** Let us see how pentagon numbers appear in counting of partitions of n. First we need to learn how to multiply polynomial expressions. Let z be a formal variable (a letter, a symbol, you can substitute any number instead of it) which we can multiply by itself and by integers and add or subtract those expressions:

$$z,\ 1+z,\ 1-z,\ 5z,\ z^2,\ 5z^4$$

Then consider the following product (we'll ignore $\times$ in the future)

$$(1-z)(1-z^2) = 1\times(1-z^2)-z\times(1-z^2) = 1\times1-1\times z^2-z\times1-z\times(-z^2) = 1-z-z^2+z^3$$

One can continue

$$(1-z)(1-z^2)(1-z^3) = (1-z-z^2+z^3)(1-z^3) = (1-z-z^2+z^3-z^3)1+(1-z-z^2+z^3)(-z^3)$$
$$= 1-z-z^2+z^3-z^3-z^3+z^4+z^5-z^6 = 1-z-z^2+z^4+z^5-z^6$$

Notice that z^3 in the last calculation got cancelled.

Problem: If you understood how to multiply polynomials calculate

$$(1-z)(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)(1-z^7)(1-z^8)$$

Do you see any cancelations? Which terms remain? Do you see a pattern?

We can keep multiplying to infinity

$$\phi(z) = \prod_{k=1}^{\infty}(1-z^k) = (1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots$$

we denoted this expression (function $\phi(z)$). If you have done the above problem then you saw that

$$\phi(z) = 1-z^{1}-z^{2}+z^{5}+z^{7}-z^{12}-z^{15}+z^{22}+z^{26}-\ldots$$

Our favorite pentagon numbers again! Now they appear as powers of z. Also notice familiar signs appearing in pairs.

8.2. **Generating Function for $p(n)$.** Now let us consider the following expression which is the inverse of $\phi(z)$

$$\mathbf{p}(z) = \frac{1}{\phi(z)} = \prod_{k=1}^{\infty}\frac{1}{1-z^k} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots} \tag{11}$$

it may look at little scary at the moment, so let us break it down intro pieces. First look at a simpler expression like $\frac{1}{1-z}$. First we show that

$$\frac{1}{1-z} = 1+z+z^2+z^3+z^4+\ldots \tag{12}$$

This follows from the following. Fix integer N then consider

$$1 - z^N = (1-z)(1 + z + z^2 + z^3 + \cdots + z^{N-1}) \tag{13}$$

Problem: Verify the above equation for $N = 5$, namely:

$$1 - z^5 = (1-z)(1 + z + z^2 + z^3 + z^4)$$

Check this by expanding the right hand side:$(1-z)(1+z+z^2+z^3+z^4) = 1(1+z+z^2+z^3+z^4) - z(1+z+z^2+z^3+z^4) = \ldots$

Since equation (13) holds for all N, we can make it arbitrarily large. It turns out that if z is small enough (in fact, smaller than 1) z^N is getting smaller and smaller as N is getting larger and larger. So we get

$$\frac{1-z^N}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{N-1}$$

and as N approaches infinity we get formula (12).

Next consider

$$\frac{1}{1-z^k}$$

which stands inside the product of $\mathbf{p}(z)$ in formula (11). This expression looks almost exactly like the one we saw above if we change variables as $w = z^k$, where w is another variable

$$\frac{1}{1-z^k} = \frac{1}{1-w} = 1 + w + w^2 + w^3 + w^4 + \ldots$$

Here we used formula (12) in terms of w-variable. Recalling that $w = z^k$ we get

$$\frac{1}{1-z^k} = 1 + z^k + z^{2k} + z^{3k} + z^{4k} + \ldots$$

Thus the generating function in (11) can be written as

$$\mathbf{p}(z) = \prod_{k=1}^{\infty}(1 + z^k + z^{2k} + z^{3k} + \ldots)$$

which means the following infinite product of (also infinite expressions)

$$\mathbf{p}(z) = (1 + z + z^2 + z^3 + \ldots)(1 + z^2 + z^4 + z^6 + \ldots)(1 + z^3 + z^6 + z^9 + \ldots) \cdot \ldots$$

Now we need to collect terms in front of each power of z. Each term z^n in the resulting product will look like

$$z^{k_1} \cdot z^{2k_2} \cdot z^{3k_3} \cdot \ldots \cdot z^{mk_m} = z^{k_1 + 2k_2 + 3k_3 + \cdots + mk_m}$$

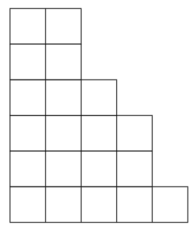
We want to count the number of such products with $k_1 + 2k_2 + 3k_3 + \cdots + mk_m = n$, that is, the number of presentations

$$n = k_1 + 2k_2 + \cdots + mk_m = \underbrace{1 + \cdots + 1}_{k_1} + \underbrace{2 + \cdots + 2}_{k_2} + \cdots + \underbrace{m + \cdots + m}_{k_m}$$

which is the number of partitions of n! This can be illustrated by looking at the Young diagram for partition

$$\{\underbrace{m,\ldots,m}_{k_m},\underbrace{m-1,\ldots,m-1}_{k_{m-1}},\ldots,\underbrace{2,\ldots,2}_{k_2},\underbrace{1,\ldots,1}_{k_1}\}$$

For example, consider partition $\{6,6,4,3,1\}$ of 20



here $m = 6$ and $k_6 = 2, k_5 = 0, k_4 = 1, k_3 = 1, k_2 = 0, k_1 = 1$. Since the powers of z in the product increase for every n there will be finitely many terms. Thus the coefficient in front of z^n in $\mathbf{p}(z)$ is equal to $p(n)$ - the number of integer partitions of n

$$\mathbf{p}(z) = 1 + p(1)z + p(2)z^2 + p(3)z^3 + p(4)z^4 +$$

The first several terms look like

$$\mathbf{p}(z) = 1+z+2z^2+3z^3+5z^4+7z^5+11z^6+15z^7+22z^8+30z^9+42z^{10}+56z^{11}+77z^{12}+101z^{13}+\ldots$$

We recognize familiar numbers $1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, \ldots$.

$\blacksquare$