Int I Farey Sequence II 5/5/21

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Farey Sequence II

BMC Int I Spring 2021

~~April 28~~, 2021 May 5

1 Pick's Theorem Applications

Theorem 1.1. *Let S be a polygon whose vertices all occur at lattice points. Let A be its area, I be the number of interior points and P be the number of points on the perimeter. Then $A = I + P/2 - 1$.*

Exercise 1.2. *Prove that you cannot draw an equilateral triangle so that all of its vertices are lattice points.*

Exercise 1.3. *Prove that you cannot draw a regular hexagon so that all of its vertices are lattice points.*

Exercise 1.4. *What is the smallest area of a convex pentagon that has all its vertices at lattice points?* (Putnam (AMC for college students))

Exercise 1.5. *Consider the tetrahedron in 3D with vertices at $(0,0,0), (1,0,0), (0,1,0), (1,1,r)$. What is its volume? How many lattice points are in the interior? What about on the boundary?*

2 Ford Circles

Definition 2.1. *A Ford circle is a circle whose center is at $(\frac{p}{q}, \frac{1}{2q^2})$ and whose radius is $\frac{1}{2q^2}$.*

Exercise 2.2. *Draw the Ford circles corresponding to $0/1, 1/1$ and $1/2$. What do you notice about them?*

Exercise 2.3. *Find the radius and center of the circle that is tangent to both the Ford circles $0/1, 1/2$ and the x axis.*

Exercise 2.4. *Do you notice anything? Can you prove any conjecture that you have?*

Exercise 2.5. *Prove that two Ford circles are either tangent to each other or don't touch each other at all.*

Exercise 2.6. *Geometrically prove that if α is irrational, then there are infinitely many fractions p/q such that $|\alpha - p/q| < \frac{1}{2q^2}$.*

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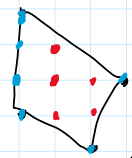
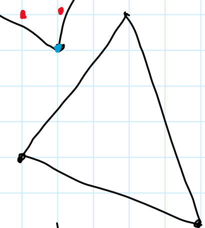
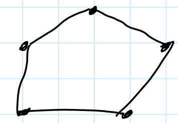
points inside ↓ points on perim. ↓

$A = I + \frac{P}{2} - 1$

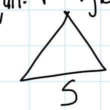
$= 5 + \frac{6}{2} - 1 = 7$







equil. triangle



Area $= \frac{5^2\sqrt{3}}{4}$

Proof by Contradiction

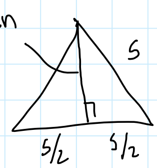
Assume the opposite is true and then derive a mathematical / logical impossibility

12) Suppose for the sake of contradiction that we can draw an equil. triangle

By Pick's Theorem, the area is of the form $I + \frac{P}{2} - 1$ so it looks like n or $n\frac{1}{2}$ for some integer n.

The area of an equilateral triangle is $\frac{5^2\sqrt{3}}{4} = n$ or $n\frac{1}{2}$ $\Rightarrow$ $\sqrt{3}$ can be written as $\frac{p}{q}$ a fraction.

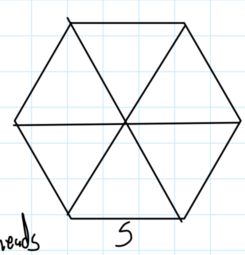
Pythagorean $\Rightarrow \frac{5\sqrt{3}}{2}$



Contradiction ↯ ← But $\sqrt{3}$ is irrational (can't be written as a fraction)

Therefore, we cannot draw an equil. triangle

1.3

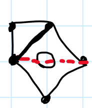


Area $= \frac{6\,5^2\sqrt{3}}{4}$

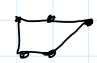
Area hexagon $= 6 \cdot$ Area of equil. triangle $= \frac{6\,5^2\sqrt{3}}{4}$

1.4 Convex pentagon

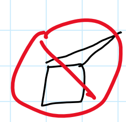
no arrowheads "



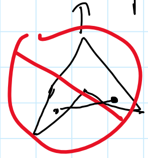
Area $= I + \frac{P}{2} - 1 = 1 + \frac{5}{2} - 1 = 2.5$



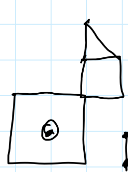
Can we get any smaller?

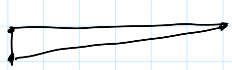


not convex



← not convex





For any pentagon, $P \geq 5$

Can we also have $I = 0$?

Suppose the 5 vertex points are $(x_1, y_1) \cdots (x_5, y_5)$.

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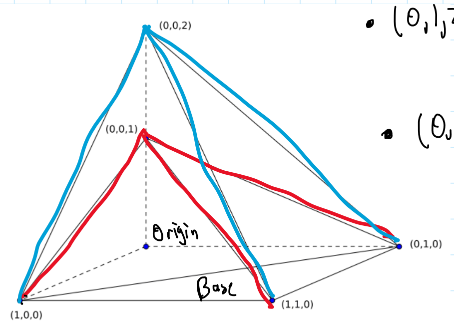
They could be (even,even), (even,odd), (odd, even), (odd, odd)

There are 5 points and 4 parities $\Rightarrow$ 2 points have the same parity $\Rightarrow$ midpoint is a lattice point = interior point $\Rightarrow I \geq 1$



• $(0,1,2)$

• $(0,1,1)$

Base is a triangle of area $\frac{1}{2}$

Red Tetrahedron Volume $= \frac{B \cdot h}{3} = \frac{\frac{1}{2} \cdot 1}{3} = \frac{1}{6}$

$I = 0$
$P = 4$

Blue Tetrahedron Volume $= \frac{B \cdot h}{3} = \frac{\frac{1}{2} \cdot 2}{3} = \frac{1}{3}$

$I = 0$
$P = 4$

$\text{Area} = \underbrace{I}_{\text{integer}} + \underbrace{\frac{P}{2}}_{\text{integer}} - 1$

There isn't a Pick's Theorem in 3D

Definition 2.1. *A Ford circle is a circle whose center is at* $(\frac{p}{q}, \frac{1}{2q^2})$ *and whose radius is* $\frac{1}{2q^2}$.

Exercise 2.2. *Draw the Ford circles corresponding to* $0/1, 1/1$ *and* $1/2$. *What do you notice about them?*

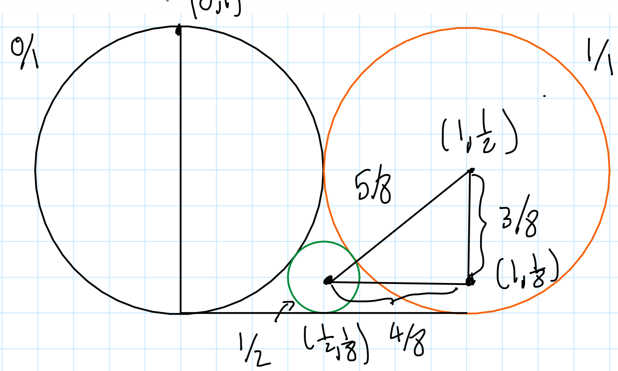
Center $= \left(\frac{1}{2}, \frac{1}{8}\right)$ radius $= \frac{1}{8}$

Exercise 2.3. *Find the radius and center of the circle that is tangent to both the Ford circles* $0/1, 1/2$ *and the* x *axis.*

Exercise 2.4. *Do you notice anything? Can you prove any conjecture that you have?*

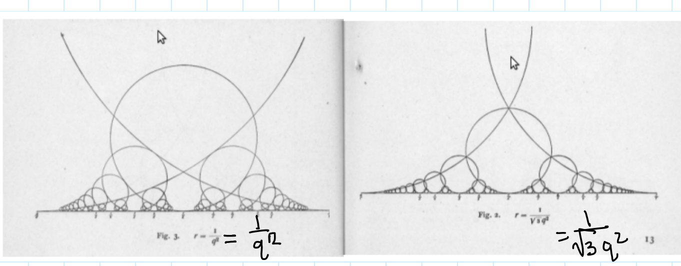
Exercise 2.5. *Prove that two Ford circles are either tangent to each other or don't touch each other at all.*

Exercise 2.6. *Geometrically prove that if* α *is irrational, then there are infinitely many fractions* p/q *such that* $|\alpha - p/q| < \frac{1}{2q^2}$.

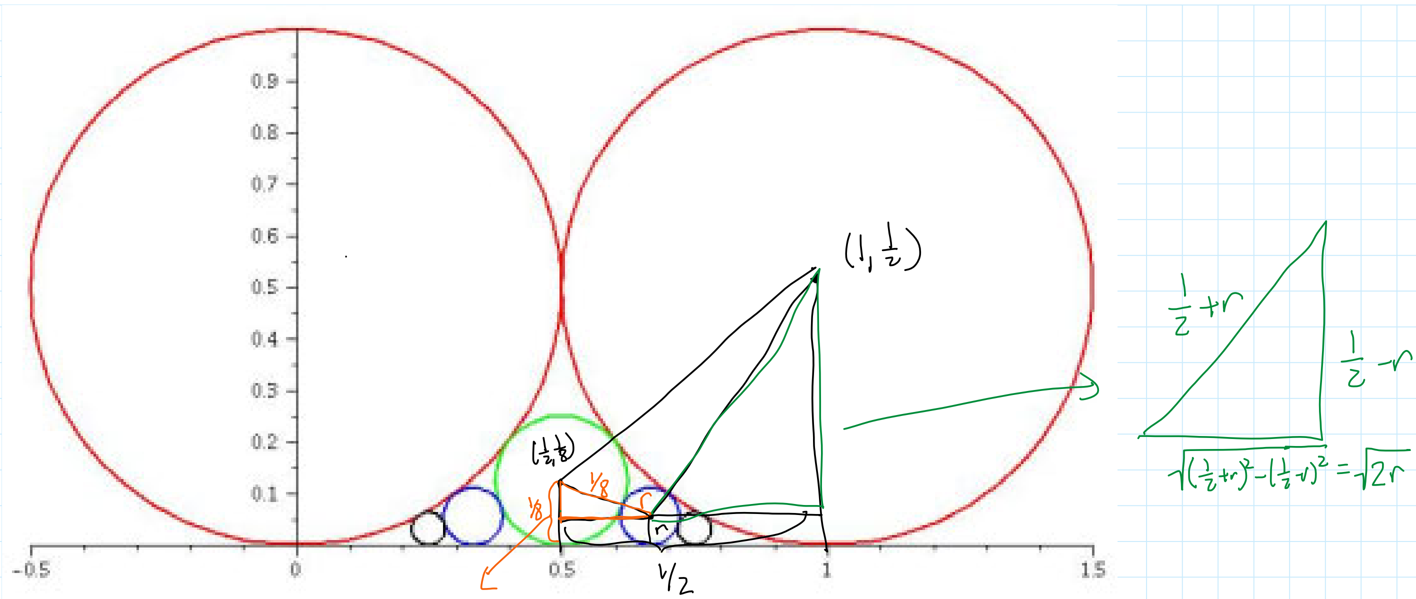


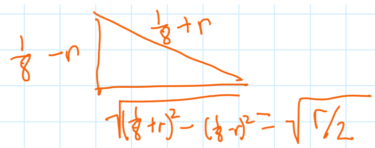
radius of orange

$\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$ radius of green



When we set $r = \frac{1}{2q^2} \Rightarrow$ circles become tangent





$\sqrt{r/2}+\sqrt{2r}=\frac{1}{2} \Rightarrow r=\frac{1}{18}$

Center of the circle is $(\frac{2}{3}, \frac{1}{18})$

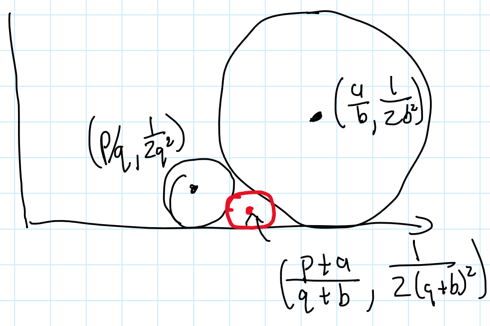
Ford circle corresponding to $\frac{2}{3}$: circle centered at $\left(\frac{2}{3}, \frac{1}{2\cdot 3^2}\right)$ with radius $\frac{1}{2\cdot 3^2}=\frac{1}{18}$

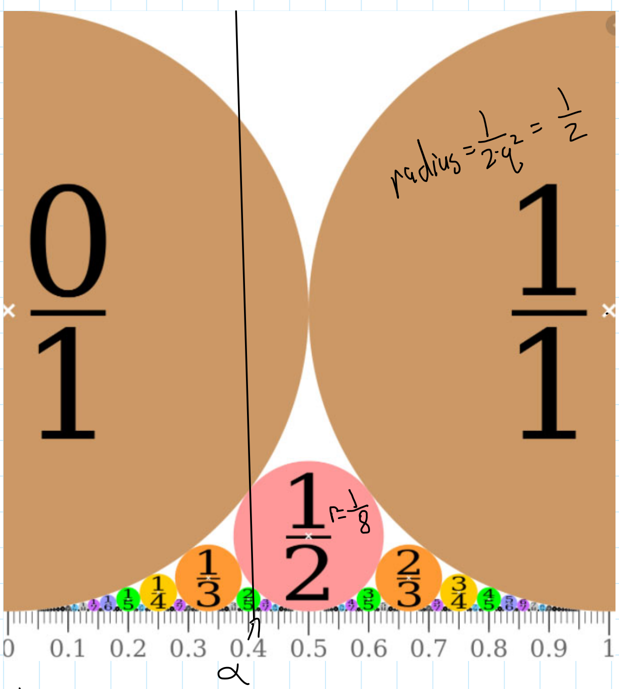
$\Rightarrow$ Ford Circle tangent to $\frac{1}{2}$ and $\frac{1}{1}$ is $\frac{2}{3}=\frac{1+1}{2+1}$

—— " —— $\frac{0}{1}$ and $\frac{1}{2}$ is $\frac{1}{3}=\frac{0+1}{1+2}$

—— " —— $\frac{0}{1}$ and $\frac{1}{1}$ is $\frac{1}{2}=\frac{0+1}{1+1}$

We can prove this is true in general





Prove that any α has ∞-ly many fractions p/q s.t.

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{2q^2}$$

e.g. $\alpha = 0.41$

If the line intersects a circle $(\frac{p}{q}, \frac{1}{2q^2})$.

That means difference in x-coor. < radius

$$\underbrace{\text{difference in x-coor.}}_{\left|\frac{p}{q}-\alpha\right|}<\frac{1}{2q^2}$$

This line intersects ∞'ly many circles.

There is a connection between Farey Sequence and Riemann Hypothesis

α

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Famous unsolved math problem

"Millenium problem"

$F_3 = \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$ length $= L_3 = 5$

$F_4 = \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$ length $= L_4 = 7$

How much does the Farey seq. differ from a uniform dist.?

Compute $\left|\frac{0}{1} - \frac{1}{5}\right| + \left|\frac{1}{3} - \frac{2}{5}\right| + \left|\frac{1}{2} - \frac{3}{5}\right| + \left|\frac{2}{3} - \frac{4}{5}\right| + \left|\frac{1}{1} - \frac{5}{5}\right| = D_3$

and $\left|\frac{0}{1} - \frac{1}{7}\right| + \left|\frac{1}{4} - \frac{2}{7}\right| + \cdots + \left|\frac{1}{1} - \frac{7}{7}\right| = D_4$

How fast does D_n grow?

Proving the Riemann Hypothesis $\Longleftrightarrow$ Proving $\frac{D_n}{n^{1/2+\varepsilon}} \to 0$ as $n \to \infty$