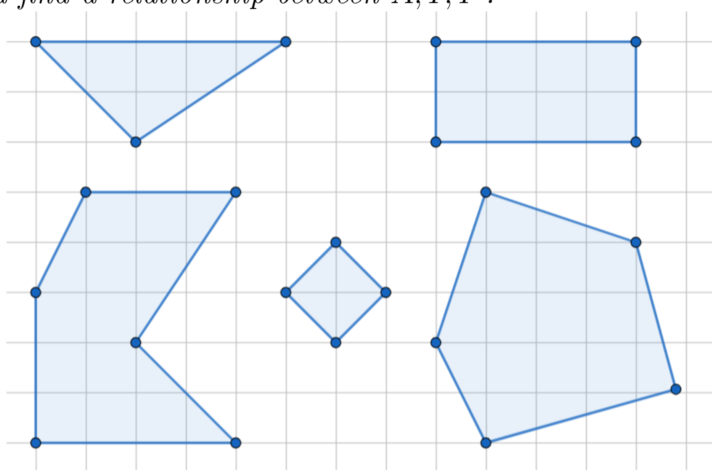
Farey Sequence I

BMC Int I Spring 2021

April 28, 2021

1 Pick's Theorem

Exercise 1.1. *For each of the following figures, find the area A. Also, record the number of lattice points on the interior of the shape I, and the number of lattice points on the perimeter P. Can you find a relationship between A, I, P?*



Theorem 1.2 (Pick's Theorem)**.** *Let S be a polygonal shape whose vertices are at lattice points. Then the area A, number of interior points I, and perimeter points P satisfy a relation:*

$$A =$$

Example 1.3. *Prove that if two polygons S_1, S_2 satisfy Pick's Theorem and they share an edge, then their "sum" also satisfies Pick's Theorem.*

Exercise 1.4. *Prove that any rectangle with side lengths parallel to the x and y axis satisfies Pick's Theorem.*

Exercise 1.5. *Prove that any right triangle with legs parallel to the x and y axis satisfies Pick's Theorem.*

Exercise 1.6. *Prove that any triangle satisfies Pick's Theorem.*

Exercise 1.7. *Prove that any polygonal shape satisfies Pick's Theorem.*

Exercise 1.8. *Can you think of any generalizations of Pick's Theorem? What if the polygon has holes in it? What if we look at an equilateral triangle grid?*

2 Farey Sequence

Definition 2.1. *The Farey Sequence F_n of fractions is made up of fractions $\frac{p}{q}$ where $0 \leq p \leq q \leq n$ and $(p, q) = 1$ are coprime.*

Example 2.2. *Write the elements of F_1, F_2, F_3, F_4, F_5 in order.*

Exercise 2.3. *What patterns can you find? How many new fractions are added? Is there any pattern to the added fractions?*

Theorem 2.4. *If $\frac{a}{b} < \frac{p}{q}$ are neighbors in a Farey sequence. Then $bp - aq =$. The converse is also true.*

Corollary 2.5. *If $(p, q) = 1$ are coprime integers, then there exist x, y such that $px - qy = 1$.*

Theorem 2.6. *If $\frac{a}{b} < \frac{p}{q}$ are neighbors in a Farey sequence, then is the unique fraction in between with the smallest denominator.*

Theorem 2.7. *If $\frac{a}{b} < \frac{p}{q}$ are neighbors in a Farey sequence F_n, then*

$$-\frac{a}{b} \leq \frac{1}{b(n+1)}$$

and

$$\frac{p}{q} - \qquad \leq \frac{1}{q(n+1)}.$$

Theorem 2.8 (Dirichlet's Approximation Theorem)**.** *For any real number $\alpha \in [0, 1]$, there are infinitely many fractions $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Theorem 2.9 (Hurwitz's Theorem)**.** *For any real number $\alpha \in [0, 1]$, there are infinitely many fractions $\frac{p}{q}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2\sqrt{5}}$$