Theorem 1. The number of trees on \( n \) labeled vertices is \( n^{n-2} \).

Exercise 2. Verify this for \( n \) from 1 to 5.

1 Double Counting Proof

We count the number of ways to add directed edges to the empty graph until they form a rooted tree in two ways.

Exercise 3. Find the number of ways to choose an edge to add after \( m \) edges have already been added.

Exercise 4. Find the final count in terms of the number of rooted trees.

2 Birooted Tree Proof

Definition 5. A birooted tree is a triple \((T, i, j)\) for \( T \) a tree and \( i, j \in V(T) \).

We construct a bijection between birooted trees and functions from \([n]\) to \([n]\). Let \( P \) be the path between \( i \) and \( j \). Map every vertex \( v \notin P \) to the next vertex on the path from \( v \) to \( i \). Map the vertices of \( P \) according to their permutation. (For example, if \( P \) were \((5, 1, 3, 6)\), 1, 3, 5, and 6 would map to 5, 1, 3, and 6, respectively.)

Exercise 6. Find the image of \((T, 5, 6)\).

Theorem 7. This is a bijection.

3 Matrix Tree Theorem Proof

Definition 8. \( t(G) \) is the number of spanning trees of a graph \( G \).

Definition 9. Given an edge \( e \) of a graph \( G \), we denote the removal of \( e \) from \( G \) as \( G \setminus e \), and the contraction of \( e \) as \( G/e \).

Exercise 10. Prove that \( t(G) = t(G \setminus e) + t(G/e) \)

Definition 11. The Laplacian of a graph \( G \) on \( n \) vertices is the \( n \times n \) matrix given by

\[
L_{i,j} := \begin{cases} 
\text{deg}(i) & \text{if } i = j \\
-1 & \text{if } i \neq j \text{ and } i \sim j \\
0 & \text{if } i \neq j \text{ and } i \not\sim j
\end{cases}
\]

Exercise 12. Prove that \( \det(L(G)) = \det(L(G \setminus e)) + \det(L(G/e)) \).

Theorem 13. \( t(G) = \det(L(G)) \).

Exercise 14. Evaluate \( t(K_n) \).
4 Prüfer Sequence Proof

**Definition 15.** Given a tree, its Prüfer sequence is generated by repeatedly picking the leaf with the smallest label, writing down its neighbor, and removing it, until there are only two vertices left.

**Exercise 16.** Find the Prüfer sequence of $T$.

**Exercise 17.** Find the tree with Prüfer sequence $(3, 7, 4, 8, 2, 4)$.

**Theorem 18.** The function from trees to Prüfer sequences is a bijection.

5 Parking Function Proof

Let $f : [n] \to [n]$ be a function. $n$ cars come into a parking lot. The $j$th car drives up to and parks at the $f(j)$th space if available, otherwise it parks at the next available parking space after that.

**Definition 19.** $f$ is called a parking function if every vehicle can claim a parking spot.

**Exercise 20.** Is $(6, 4, 1, 4, 2, 6, 4)$ a parking function?

**Exercise 21.** Is $(3, 1, 5, 2, 1, 7, 5)$ a parking function?

**Theorem 22.** There are $(n+1)^{n-1}$ parking functions.

**Theorem 23.** There is a bijection between parking functions and rooted forests with $n+1$ vertices.