Cayley's Formula

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February 3, 2021

Theorem 1. *The number of trees on n labeled vertices is n^{n-2}.*

Exercise 2. Verify this for n from 1 to 5.

1 Double Counting Proof

We count the number of ways to add directed edges to the empty graph until they form a rooted tree in two ways.

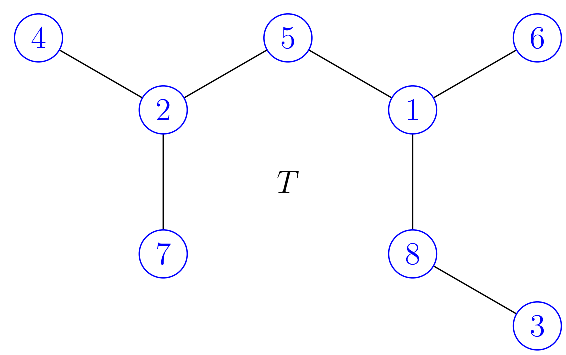
Exercise 3. Find the number of ways to choose an edge to add after m edges have already been added.

Exercise 4. Find the final count in terms of the number of rooted trees.

2 Birooted Tree Proof

Definition 5. A *biroted tree* is a triple (T, i, j) for T a tree and $i, j \in V(T)$.

We construct a bijection between birooted trees and functions from $[n]$ to $[n]$. Let P be the path between i and j. Map every vertex $v \notin P$ to the next vertex on the path from v to i. Map the vertices of P according to their permutation. (For example, if P were $(5, 1, 3, 6)$, $1, 3, 5$, and 6 would map to $5, 1, 3$, and 6, respectively.)



Exercise 6. Find the image of $(T, 5, 6)$.

Theorem 7. *This is a bijection.*

3 Matrix Tree Theorem Proof

Definition 8. $t(G)$ is the number of spanning trees of a graph G.

Definition 9. Given an edge e of a graph G, we denote the removal of e from G as $G \backslash e$, and the contraction of e as G/e.

Exercise 10. Prove that $t(G) = t(G \backslash e) + t(G/e)$

Definition 11. The Laplacian of a graph G on n vertices is the $n \times n$ matrix given by

$$L_{i,j} := \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{if } i \neq j \text{ and } i \not\sim j \end{cases}$$

Exercise 12. Prove that $\det(L(G)) = \det(L(G \backslash e)) + \det(L(G/e))$.

Theorem 13. $t(G) = \det(L(G))$.

Exercise 14. Evaluate $t(K_n)$.

4 Prüfer Sequence Proof

Definition 15. Given a tree, its *Prüfer sequence* is generated by repeatedly picking the leaf with the smallest label, writing down its neighbor, and removing it, until there are only two vertices left.

Exercise 16. Find the Prüfer sequence of T.

Exercise 17. Find the tree with Prüfer sequence $(3, 7, 4, 8, 2, 4)$.

Theorem 18. *The function from trees to Prüfer sequences is a bijection.*

5 Parking Function Proof

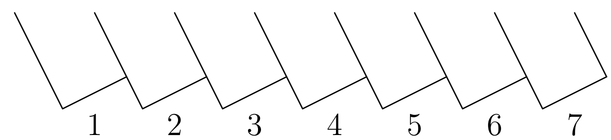
Let $f : [n] \to [n]$ be a function. n cars come into a parking lot. The j^{th} car drives up to and parks at the $f(j)^{\text{th}}$ space if available, otherwise it parks at the next available parking space after that.

Definition 19. f is called a *parking function* if every vehicle can claim a parking spot.

Exercise 20. Is $(6, 4, 1, 4, 2, 6, 4)$ a parking function?

Exercise 21. Is $(3, 1, 5, 2, 1, 7, 5)$ a parking function?

Theorem 22. *There are* $(n + 1)^{n-1}$ *parking functions.*



Theorem 23. *There is a bijection between parking functions and rooted forests with* $n + 1$ *vertices.*