Berkeley Math Circle: Intermediate II **Bases Part II**

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1. 121212121212_3 = what number in base 9?

2. A recently discovered prime is $2^{74,207,281} - 1 = x$. Find the sum of digits when x is written in base 2.

3. What is the greatest four digit perfect square in base 7?

4. $a = \overline{xyz}_9 = \overline{zyx}_6$. Find $x + y + z$.

5. Let b be an integer and $b > 2$. Let $N_b = 1_b + 2_b + ... + 100_b$. (The sum contains all valid base-b numbers up to 100_b.) Compute the number of values of b for which the sum of the squares of the base-b digits of N_b is at most 512.

6. The increasing sequence $1, 3, 4, 9, 10, 12, 13, ...$ consists of all positive integers which are powers of 3 or sum of distinct powers of 3. Find the 100^{th} term of this sequence.

7. a) Suppose $P(x)$ is an unknown polynomial, of unknown degree, with non-negative integer coefficients. Your goal is to determine this polynomial. You have access to an oracle that, given an integer n, spits out $P(n)$, the value of the polynomial at n. However, the oracle charges a fee for each such computation, so you want to minimize the number of computations you ask the oracle to do. Show that it is possible to uniquely determine the polynomial after only two consultations of the oracle.

b) Let f be a polynomial with non-negative integer coefficients. If $f(1) = 7$ and $f(7) = 7597$, what is $f(10)$?

1977 Canadian Math Olympiad Problem 3

8. Using a weighing balance, on which weights can be placed on both sides, what is the minimum number of weights one needs to be able to measure all the integral weights between 1 and 1000?
e.g. With weights 1 and 3, one could measure not only $1, 3, 1 + 3 = 4$, but also 2 by placing weights 1 and 3 on the opposite sides.

9. A weird calculator has a numerical display and only two buttons, F and G. The first button doubles the displayed number and then adds 1. The second button doubles the displayed number and then subtracts 1. For example, if the display is showing 5, then pressing F produces 11, but pressing G would produce 9. If the display shows 5 and we press the sequence F G F F, we get 87.

i) Suppose the initial displayed number is 1. Give a sequence of exactly eight button presses that will result in a https://www.overleaf.com/project/5c43a5a31f88926e9a5583d5display of 313.

ii) Suppose the initial displayed number is 1, and we then perform exactly eight button presses. Describe all the numbers that can possibly result. Prove your answer by explaining how all of these numbers can be produced and no other numbers can be produced.

10. Let N be the number of positive integers that are less than or equal to 2003 and whose base-2 representation has more 1's than 0's. Find the remainder when N is divided by 1000.

11. How many trailing zeros does the number $(2^{16})!$ have in base-2 representation?

Problems taken from AMC, AIME, Brilliant.org, AoPS and BAMO.

Supplementary problem with an original observation (not related to Bases):
n had 10 positive divisors.
$2n$ has 15 positive divisors.
$3n$ has 20 positive divisors.
How many positive divisors does $4n$ have?

While solving this problem, I made an interesting observation:
If $f(m)$ = number of (positive) divisors of m, then for a given integer n and a given prime power $m = p^r$, $f(nm^{k+1}) - f(nm^k) = f(nm) - f(n)$ for all positive integers k In addition, the difference is divisible by r.