Farey Sequence I

BMC Int I Spring 2021

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1 Pick's Theorem Applications

Theorem 1.1. *Let S be a polygon whose vertices all occur at lattice points. Let A be its area, I be the number of interior points and P be the number of points on the perimeter. Then $A = I + P/2 + 1$.*

Exercise 1.2. *Prove that you cannot draw an equilateral triangle so that all of its vertices are lattice points.*

Exercise 1.3. *Prove that you cannot draw a regular hexagon so that all of its vertices are lattice points.*

Exercise 1.4. *What is the smallest area of a convex pentagon that has all its vertices at lattice points?*

Exercise 1.5. *Consider the tetrahedron in 3D with vertices at $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, r)$. What is its volume? How many lattice points are in the interior? What about on the boundary?*

2 Ford Circles

Definition 2.1. *A Ford circle is a circle whose center is at $(\frac{p}{q}, \frac{1}{2q^2})$ and whose radius is $\frac{1}{2q^2}$.*

Exercise 2.2. *Draw the Ford circles corresponding to $0/1, 1/1$ and $1/2$. What do you notice about them?*

Exercise 2.3. *Find the radius and center of the circle that is tangent to both the Ford circles $0/1, 1/2$ and the x axis.*

Exercise 2.4. *Do you notice anything? Can you prove any conjecture that you have?*

Exercise 2.5. *Prove that two Ford circles are either tangent to each other or don't touch each other at all.*

Exercise 2.6. *Geometrically prove that if α is irrational, then there are infinitely many fractions p/q such that $|\alpha - p/q| < \frac{1}{2q^2}$.*

3 Miscellaneous

Exercise 3.1. *Sum all the fractions in the Farey sequences with denominators up to* n*,* F_n*. What do you notice?*

Remark. *The Farey sequence is connected to the elusive Riemann Hypothesis.*

Example 3.2. *Let* $\alpha \in \mathbb{R}$ *be an arbitrary number. There are infinitely many coprime integer* p, q *such that*

$$|q\alpha - p| < \frac{1}{q}.$$

Definition 3.3. *An algebraic number is a number that is a solution to a polynomial equation.*

Theorem 3.4. *For any* $\epsilon > 0$ *and any algebraic number* α*, for all rational number* p/q*, there are only finitely many solutions to*

$$|\alpha - p/q| \leq \frac{1}{q^{2+\epsilon}}.$$

Corollary 3.5. *Transcendental numbers (numbers which are not solutions to polynomial equations) exist.*