From Polygons to String Theory

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In this article we introduce a special kind of polygon called a reflexive polygon, and higher-dimensional generalizations called reflexive polytopes. In two dimensions, reflexive polygons are also called Fano polygons, after Gino Fano, an Italian mathematician born in 1871 who studied the relationship between geometry and modern algebra.

Two theoretical physicists, Maximilian Kreuzer and Harald Skarke, worked out a detailed description of three- and four-dimensional reflexive polytopes in the late 1990s. Why were physicists studying these polytopes? Their motivation came from string theory. Physicists use these polytopes to construct Calabi-Yau manifolds, which are geometric spaces that can model extra dimensions of our universe. A simple relationship between "mirror pairs" of polytopes corresponds to an extremely subtle connection between pairs of these geometric spaces. The quest to understand this connection has created the thriving field of mathematical research known as mirror symmetry.

What are reflexive polytopes? What does it mean to classify them? Why do they come in pairs? How can we build a complicated geometric space from a simple object like a triangle or a cube? And what does any of this have to do with physics? By answering these questions, we will uncover intricate relationships between combinatorics, geometry, and modern physics.

Classifying reflexive polygons

The points in the plane $\mathbb{R}^2$ with integer coordinates form a lattice, which we'll name N. A *lattice polygon* is a polygon whose vertices are in the lattice; in other words, lattice polygons have vertices with integer coordinates. We consider only convex polygons. An example of a convex lattice polygon is in FIGURE 1.

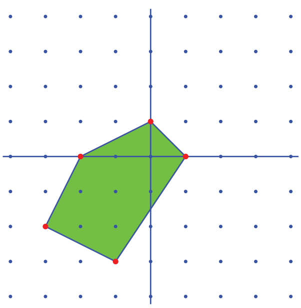

Figure 1 A lattice polygon

We say a lattice polygon is a *Fano polygon* if it has only one lattice point, the origin, in its interior.

How many Fano polygons are there? Can we list them all? The first step is to pull out our graph paper and try to draw a Fano polygon. A little experimentation will produce several Fano polygons, including triangles, quadrilaterals, and hexagons. Some examples are shown in FIGURE 2.

There are also ways to make new Fano polygons, once we find our first Fano polygon. For instance, we may rotate by 90 degrees or reflect across the x-axis. A more

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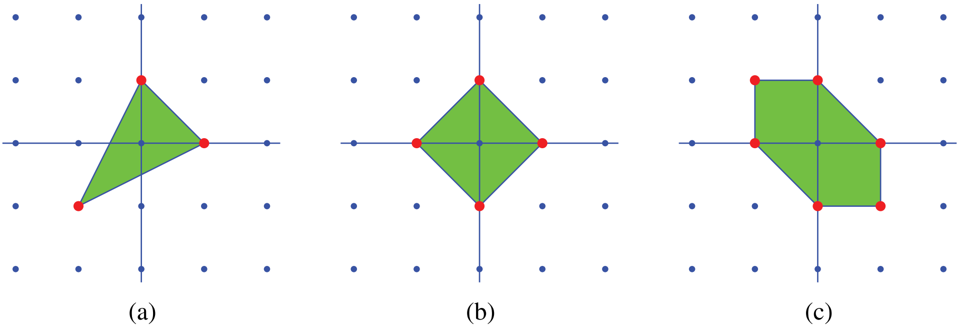

Figure 2 Fano triangle, quadrilateral, and hexagon

complicated type of map is the shear, which stretches a polygon in one direction. We can describe the shear using matrix multiplication: We map the point $\binom{x}{y}$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}.$$

In FIGURE 3 we illustrate the effects of this shear on a Fano triangle. Notice that after the shear, in FIGURE 3(b), there is still only one point in the interior of our triangle. Repeating the shear map, as seen in FIGURES 3(c) and 3(d), makes our triangle longer and skinnier. Iterating the shear map produces an infinite family of Fano triangles, each one longer and skinnier than the one before it.

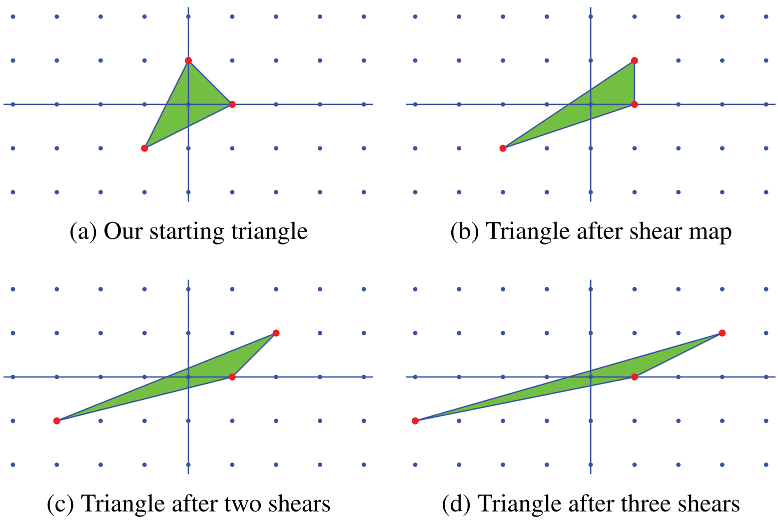

Figure 3 Effects of shears

This means that listing all Fano polygons is an impossible task! But we would still like to classify Fano polygons in some way. To do so, we shift our focus: Instead of counting individual Fano polygons, we will count types or classes of Fano polygons. We want two Fano polygons to belong to the same equivalence class if we can get from one to the other using reflections, rotations, and shears. Each map that we use should send lattice polygons to other lattice polygons.

Reflections, rotations, shears, and compositions of reflections, rotations, and shears are all linear transformations of the plane. In other words, they can be described by two-by-two matrices. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix, where a, b, c, and d are real numbers. We let m_A be the "multiplication by A" map:

$$m_A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What conditions do we need to place on the matrix A to ensure that the corresponding map m_A sends Fano polygons to other Fano polygons? First, the map must send polygons to polygons, rather than squashing them into line segments. This means that A must be invertible, so that m_A is a one-to-one map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Furthermore, m_A must send points with integer coordinates to other points with integer coordinates, so that lattice polygons go to lattice polygons. This will happen when A has integer entries.

The final condition on A is more subtle. We want two Fano polygons to belong to the same equivalence class if we can get from one to the other using rotations, reflections, and shears that map lattice polygons to lattice polygons. But our notion of equivalence should treat all Fano polygons equally: If two polygons $\diamond$ and $\diamond'$ are equivalent, it shouldn't matter whether we started with $\diamond$ and rotated or sheared it to form $\diamond'$, or started with $\diamond'$ and rotated or sheared it back to $\diamond$. In other words, our notion of equivalence must be symmetric. Now, if m_A sends $\diamond$ to $\diamond'$, the map that sends $\diamond'$ back to $\diamond$ is the map $m_{A^{-1}}$ defined by the inverse matrix A^{-1}. To ensure that $m_{A^{-1}}$ sends lattice polygons to lattice polygons, we require A^{-1} to have integer entries.

We can characterize A and A^{-1} using determinants. Because the product $A\,A^{-1}$ is the identity matrix, $(\det A)(\det A^{-1}) = 1$. But the determinant of a matrix with integer entries is an integer, so $\det A$ and $\det A^{-1}$ are both integers. The only way two integers can multiply to give 1 is for them to be both 1 or both -1, so $\det A = \pm 1$.

Matrices that satisfy this determinant property have a special name.

DEFINITION. $\mathbf{GL}(2, \mathbb{Z})$ is the set of two-by-two matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, which have integer entries and determinant $ad - bc$ equal to either 1 or -1.

The matrices in $\mathbf{GL}(2, \mathbb{Z})$ form a group, which is generated by rotation matrices, reflection matrices, and the shear matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, the matrices in $\mathbf{GL}(2, \mathbb{Z})$ describe maps that are rotations, reflections, shears, or compositions of rotations, reflections, or shears.

By construction, for any matrix A in $\mathbf{GL}(2, \mathbb{Z})$, the map m_A is a continuous, one-to-one, and onto map of the plane, which restricts to a one-to-one and onto map from the lattice N to itself. The map sends lattice polygons to lattice polygons. In particular, the origin is the only lattice point in the interior of a Fano polygon, so it is the only lattice point that can be mapped to the interior of the image of a Fano polygon. Thus, this map sends Fano polygons to Fano polygons.

DEFINITION. We say two Fano polygons Δ and Δ' are $\mathbf{GL}(2, \mathbb{Z})$*-equivalent* (or sometimes just *equivalent*) if there exists a matrix A in $\mathbf{GL}(2, \mathbb{Z})$ such that $m_A(\Delta) = \Delta'$.

FIGURE 4 shows an example of equivalent polygons.

Now that we have a concept of equivalent Fano polygons, we can try again to describe the possible Fano polygons. How many $\mathbf{GL}(2, \mathbb{Z})$ equivalence classes of Fano polygons are there?

It turns out that there are only 16 equivalence classes of Fano polygons! A representative from each Fano polygon equivalence class is shown in FIGURE 5. (The figure seems to show twenty classes, but four of them are duplicated, for reasons that we will

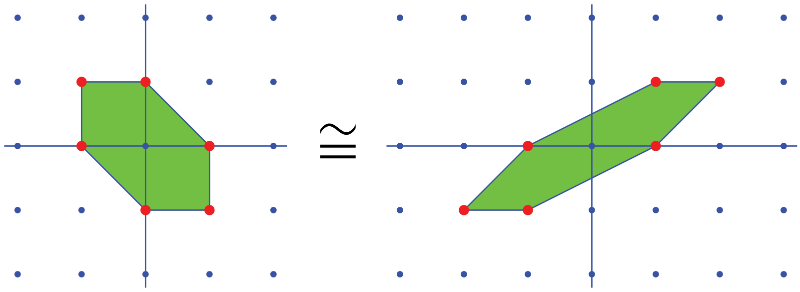

Figure 4 Two equivalent hexagons

investigate in the next section.) We will use the classification of Fano polygons later, but here we omit the proof; a combinatorial proof may be found in Nill [**7**].

Polar polygons The vertical arrows in FIGURE 5 indicate relationships between pairs of Fano polygons. In this section, we explain the correspondence. We start by asking a simple question: How can we describe a Fano polygon mathematically?

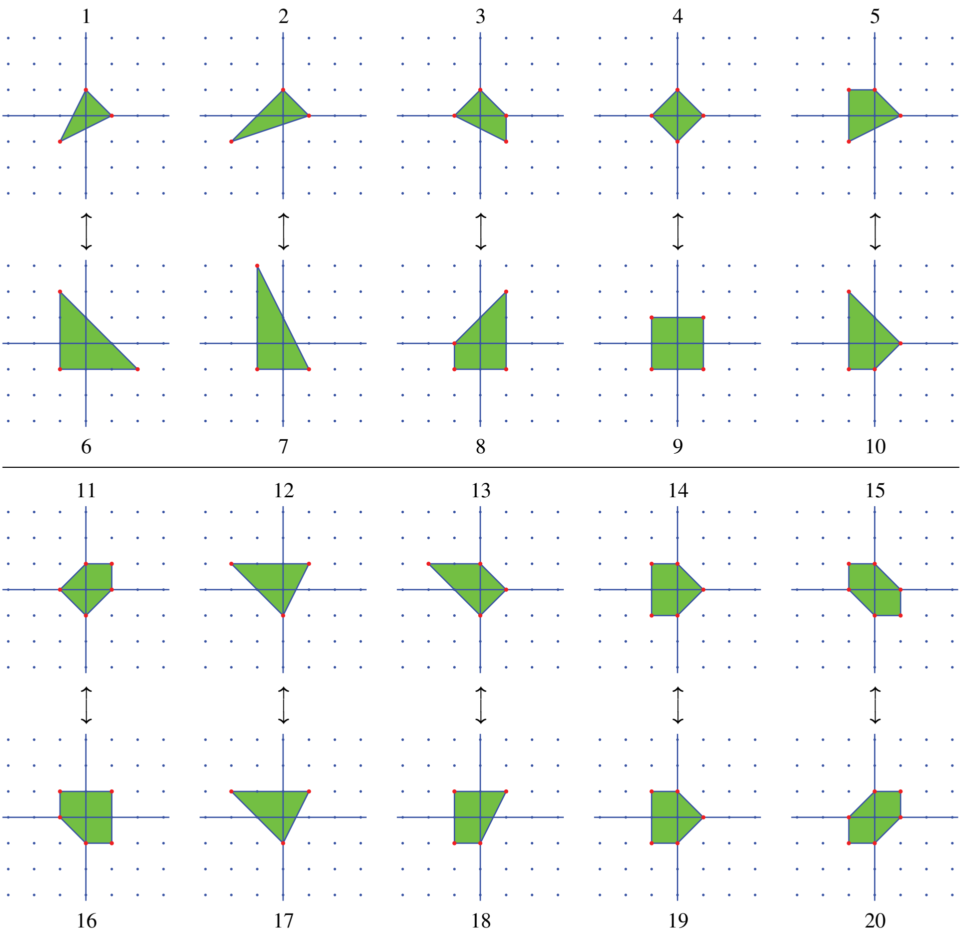

Figure 5 Classification of Fano polygons

One way to describe a polygon is to list the vertices. For instance, for the triangle in FIGURE 2(a), the vertices are $(0, 1)$, $(1, 0)$, and $(-1, -1)$.

Each edge of a polygon is part of a line, so we can also describe a polygon by listing the equations of these lines. For the triangle in FIGURE 2(a), the equations are:

$$\begin{aligned} -x - y &= -1, \\ 2x - y &= -1, \\ -x + 2y &= -1. \end{aligned}$$

Of course, there are many equivalent ways to write the equation for a line. We have chosen $ax + by = -1$ as our standard form. Any line that does not pass through the origin can be written in this way. Our standard form has the advantage that the whole triangle is described as the set of points (x, y) such that

$$\begin{aligned} -x - y &\geq -1, \\ 2x - y &\geq -1, \\ -x + 2y &\geq -1. \end{aligned}$$

Notice that in the case of our Fano triangle, the coefficients a and b in our standard form for the equation of a line are all integers.

We want to use our edge equations to define a new polygon. We'd like our new polygon to live in its own copy of the plane. Let's call the set of points in this new plane that have integer coordinates M and name the new plane $M_{\mathbb{R}}$. Using the dot product, we can combine a point in our old plane, $N_{\mathbb{R}}$, with a point in our new plane, $M_{\mathbb{R}}$, to produce a real number

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2.$$

If the point (n_1, n_2) lies in N and the point (m_1, m_2) lies in M, their dot product $(n_1, n_2) \cdot (m_1, m_2)$ is an integer.

Note that every dot product in this paper combines a vector in $N_{\mathbb{R}}$ (on the left) with a vector in $M_{\mathbb{R}}$ (on the right). The lattices N and M are isomorphic, as are the planes $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, so the distinction between the two planes may seem artificial. However, as we will see later, the points in the N and M lattices play different roles in the physicists' construction.

Let's rewrite the edge equations of our Fano triangle using dot product notation:

$$\begin{aligned} (x, y) \cdot (-1, -1) &= -1, \\ (x, y) \cdot (2, -1) &= -1, \\ (x, y) \cdot (-1, 2) &= -1. \end{aligned}$$

The points $(-1, -1)$, $(2, -1)$, and $(-1, 2)$ are the vertices of a new triangle in $M_{\mathbb{R}}$. We say that the new triangle, shown in FIGURE 6, is the *polar polygon* of our original triangle.

Suppose (n_1, n_2) is any point in our original triangle, and (m_1, m_2) is any point in its polar polygon. (These points do not need to be lattice points or boundary points; they might lie in the interiors of the triangles.) If we move the points around, the dot product $(n_1, n_2) \cdot (m_1, m_2)$ will vary continuously. The minimum possible dot product is -1, as we saw from our edge equations. Thus, the dot product of the two points will

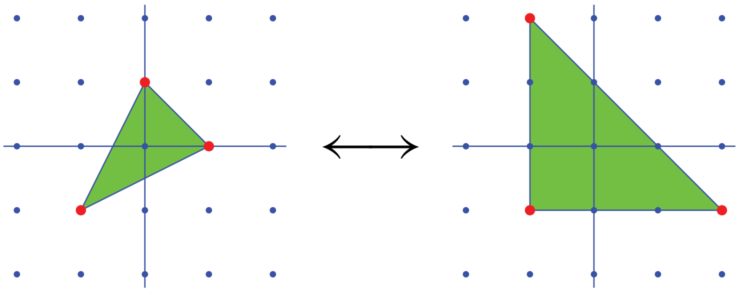

Figure 6 Our triangle and its polar polygon

satisfy the inequality

$$(n_1, n_2) \cdot (m_1, m_2) \geq -1.$$

We can use this inequality to define a polar polygon in general. Let Δ be any lattice polygon in our original plane $N_{\mathbb{R}}$ that contains $(0, 0)$. Formally, we say that the *polar polygon* Δ° is the polygon in $M_{\mathbb{R}}$ consisting of the points (m_1, m_2) such that

$$(n_1, n_2) \cdot (m_1, m_2) \geq -1$$

for all points (n_1, n_2) in Δ. We can find the vertices of Δ° using our standard-form equations for the edges of Δ, just as we did above.

The polar polygon of an arbitrary lattice polygon may fail to be a lattice polygon: The vertices might be rational numbers, rather than integers. On the other hand, the polar polygon of our example triangle is actually a lattice polygon; even better, it is itself a Fano polygon.

DEFINITION. Let Δ be a lattice polygon. If Δ° is also a lattice polygon, we say that Δ is *reflexive*.

If Δ is a reflexive polygon, then its polar polygon Δ° is also a reflexive polygon. We can repeat the polar polygon construction to find the polar polygon of Δ°. (For practice, try finding the polar polygon of the second triangle in FIGURE 6.) It turns out that the result is our original polygon:

$$(\Delta^\circ)^\circ = \Delta.$$

It's easy to see that every point in Δ is contained in $(\Delta^\circ)^\circ$, using the definition of a polar polygon. Showing that Δ and $(\Delta^\circ)^\circ$ are always identical is trickier; the proof (which we omit) depends on the fact that Δ is convex.

We say that a polygon Δ and its polar polygon Δ° are a *mirror pair*. The term *reflexive* also refers to this property: In a metaphorical sense, Δ and its polar polygon Δ° are reflections of each other. The fact that the Fano triangle in our example turned out to be reflexive is not a coincidence.

THEOREM 1. *A lattice polygon is reflexive if and only if it is Fano.*

Proof. First we show that every reflexive polygon is Fano. Let Δ be a reflexive polygon, and let (x, y) be a lattice point that lies strictly in the interior of Δ. Let $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$ be the vertices of the polar polygon Δ°; because Δ is reflexive, the coordinates of each vertex are integers. We know that

$$(x, y) \cdot (v_i, w_i) \geq -1$$

for each vertex (v_i, w_i). Because (x, y) lies in the interior of Δ, the inequality must be strict:

$$(x, y) \cdot (v_i, w_i) > -1.$$

But (x, y) and (v_i, w_i) are both lattice points, so $(x, y) \cdot (v_i, w_i)$ must be an integer. Thus,

$$(x, y) \cdot (v_i, w_i) \geq 0.$$

Now, let a be any positive integer. The point $a(x, y) = (ax, ay)$ is a lattice point, and

$$\begin{aligned} a(x, y) \cdot (v_i, w_i) &\geq a \cdot 0 \\ &\geq -1. \end{aligned}$$

Because (ax, ay) satisfies the inequalities corresponding to each vertex of Δ°, we conclude that

$$(ax, ay) \cdot (m_1, m_2) \geq -1$$

for any point (m_1, m_2) in Δ°. Therefore, (ax, ay) is a point in $(\Delta^\circ)^\circ = \Delta$. If (x, y) is not $(0, 0)$, then we get an infinite number of different lattice points in Δ, one for each positive integer a. But this is impossible: Polygons are bounded, so they cannot contain infinite numbers of lattice points! Thus, (x, y) must be $(0, 0)$, so Δ is Fano.

We will use the classification of Fano polygons illustrated in FIGURE 5 to show that every Fano polygon is reflexive. The first step is to make sure that the representatives of Fano equivalence classes shown in the figure are reflexive polygons. The vertical arrows in FIGURE 5 connect each of the illustrated polygons to its polar dual (checking this fact is a fun exercise!). The computation tells us that every Fano polygon is equivalent to a reflexive polygon. Notice that the Fano polygons numbered 12 through 15 in FIGURE 5 are self-dual: Their polar duals are equivalent to the original polygons. (Can you find the matrix that sends polygon 13 to its polar dual?)

To finish the proof, we must show that if Γ is equivalent to a reflexive polygon Δ, then Γ is also a reflexive polygon. Let Δ be a reflexive polygon, let A be a matrix in $\mathbf{GL}(2, \mathbb{Z})$, and suppose that $\Gamma = m_A(\Delta)$. Let $B = (A^{\mathrm{T}})^{-1}$, the inverse of the transpose matrix of A. Taking the transpose of a matrix does not change its determinant, and the inverse of a matrix in $\mathbf{GL}(2, \mathbb{Z})$ is also in $\mathbf{GL}(2, \mathbb{Z})$, so B is another member of $\mathbf{GL}(2, \mathbb{Z})$. We claim that $\Gamma^\circ = m_B(\Delta^\circ)$.

Let (n_1, n_2) be a point in Γ, and let (m_1, m_2) be any point in $M_\mathbb{R}$. We know that

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = A \begin{pmatrix} n_1' \\ n_2' \end{pmatrix}$$

for some point (n_1', n_2') in Δ. Let

$$\begin{pmatrix} m_1' \\ m_2' \end{pmatrix} = B^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

so that

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = B \begin{pmatrix} m_1' \\ m_2' \end{pmatrix}.$$

Now,

$$\begin{aligned}(n_1, n_2) \cdot (m_1, m_2) &= \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\ &= \left(A \begin{pmatrix} n'_1 \\ n'_2 \end{pmatrix} \right)^{\mathrm{T}} B \begin{pmatrix} m'_1 \\ m'_2 \end{pmatrix} \\ &= \begin{pmatrix} n'_1 \\ n'_2 \end{pmatrix}^{\mathrm{T}} A^{\mathrm{T}} B \begin{pmatrix} m'_1 \\ m'_2 \end{pmatrix} \\ &= \begin{pmatrix} n'_1 \\ n'_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} m'_1 \\ m'_2 \end{pmatrix} \\ &= (n'_1, n'_2) \cdot (m'_1, m'_2).\end{aligned}$$

We see that $(n_1, n_2) \cdot (m_1, m_2) \geq -1$ if and only if $(n'_1, n'_2) \cdot (m'_1, m'_2) \geq -1$, and therefore (m_1, m_2) is in Γ° if and only if (m'_1, m'_2) is in Δ°. Thus, $\Gamma^\circ = m_B(\Delta^\circ)$. Since Δ° is a lattice polygon, $m_B(\Delta^\circ)$ must be a lattice polygon, so Γ is reflexive. ■

A reflexive polygon and its polar dual are intricately related. It's pretty easy to see that a polygon and its polar dual have the same number of sides and vertices. Other connections are more subtle. For instance, the number of lattice points on the boundary of a reflexive polygon and the number of lattice points on the boundary of its polar dual always add up to twelve! For the polygons in FIGURE 6, the computation is $3 + 9 = 12$. We can check that this holds in general by counting points in FIGURE 5 and using the fact that equivalent Fano polygons have the same number of lattice points. (See [**8**] for other proofs that the boundary points add to twelve, which use combinatorics, algebraic geometry, and number theory.)

Higher dimensions

Let's extend the idea of Fano and reflexive polygons to dimensions other than 2. In order to do so, we need to describe the k-dimensional generalizations of polygons, which we will call *polytopes*. There are several ways to do this. We take the point of view that polygons are described by writing down a list of vertices, adding line segments that connect these vertices, and then filling in the interior of the polygon. Similarly, in k dimensions, our intuition suggests that we should describe a polytope by writing down a list of vertices, connecting them, and then filling in the inside. The formal definition is as follows.

DEFINITION. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ be a set of points in $\mathbb{R}^k$. The *polytope* with vertices $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ is the set of points of the form

$$\vec{x} = \sum_{i=1}^{q} t_i \vec{v}_i,$$

where the t_i are nonnegative real numbers satisfying $t_1 + t_2 + \cdots + t_q = 1$. (The polytope is called the *convex hull* of the points $\vec{v}_i$.)

We illustrate a three-dimensional polytope in FIGURE 7.

Let N be the lattice of points with integer coordinates in $\mathbb{R}^k$; we refer to this copy of $\mathbb{R}^k$ as $N_{\mathbb{R}}$. A *lattice polytope* is a polytope whose vertices lie in N.

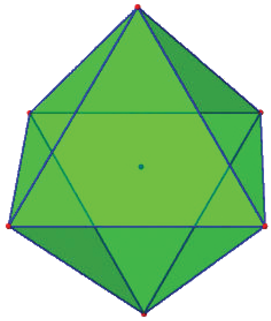

Figure 7 A three-dimensional polytope

The one-dimensional case To understand the definition of a lattice polytope better, let's consider the one-dimensional polytope that has vertices 1 and -1. Formally, this polytope consists of all points on the real number line that can be written as $1 \cdot t_1 + -1 \cdot t_2$, where $t_1 \geq 0$, $t_2 \geq 0$, and $t_1 + t_2 = 1$. We can visualize this by imagining an ant walking on the number line. Our ant starts at 0; for a fraction of an hour the ant walks to the right, toward the point 1, then for the rest of the hour the ant walks left, toward the point -1. The ant can reach any point in the closed interval $[-1, 1]$, so all of these points belong to our lattice polytope. This one-dimensional lattice polytope is shown in FIGURE 8.

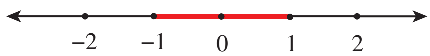

Figure 8 A one-dimensional polytope

The *k*-dimensional case Just as we did in two dimensions, we can define a *dual lattice* M in k dimensions by taking a new copy of $\mathbb{R}^k$, which we'll refer to as $M_{\mathbb{R}}$, and letting M be the points with integer coordinates in $M_{\mathbb{R}}$. The dot product pairs points in $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ to produce real numbers:

$$(n_1, \dots, n_k) \cdot (m_1, \dots, m_k) = n_1 m_1 + \cdots + n_k m_k.$$

If we take the dot product of a point in our original lattice N and a point in our dual lattice M, we obtain an integer.

We can use our k-dimensional dot product to define polar polytopes. If Δ is a lattice polytope in N that contains the origin, we say its *polar polytope* Δ° is the polytope in M given by

$$\{(m_1, \dots, m_k) : (n_1, \dots, n_k) \cdot (m_1, \dots, m_k) \geq -1 \text{ for all } (n_1, \dots, n_k) \in \Delta\}.$$

We say that a lattice polytope is *Fano* if the only lattice point that lies strictly in its interior is the origin, and that a lattice polytope containing the origin is *reflexive* if its polar polytope is also a lattice polytope. Just as in two dimensions, we find that the polar of the polar of a reflexive polytope is the original polytope ($(\Delta^\circ)^\circ = \Delta$), and we say that a reflexive polytope Δ and its polar polytope Δ° are a *mirror pair*. We illustrate a mirror pair of three-dimensional polytopes in FIGURE 9.

Every reflexive polytope is a Fano polytope: The proof that we used to show reflexive polygons are Fano carries through in k dimensions.

However, not every Fano polytope is reflexive. We can construct an example starting with the cube in FIGURE 9. The cube with vertices at $(\pm 1, \pm 1, \pm 1)$ is both Fano and

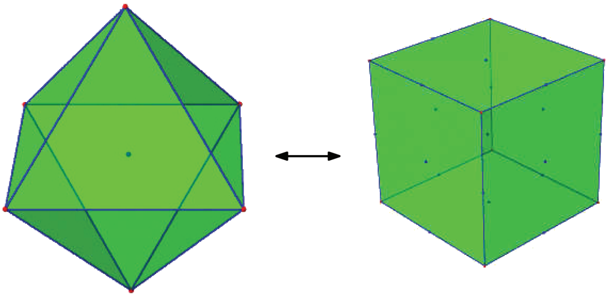

Figure 9 The octahedron and cube: a mirror pair

reflexive. But we can form a new polytope from the cube by removing the vertex at $(1, 1, 1)$ and taking the convex hull of the remaining lattice points. The resulting polytope is shown in FIGURE 10. It is clearly Fano, since the origin lies in the interior, and we have not added any lattice points. However, it is not reflexive. The equation of the face spanned by the new vertices $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$ is $x + y + z = 2$, or $-\frac{1}{2}x + -\frac{1}{2}y + -\frac{1}{2}z = -1$ in standard form. Thus, the polar polygon has $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ as a vertex, so it is not a lattice polytope.

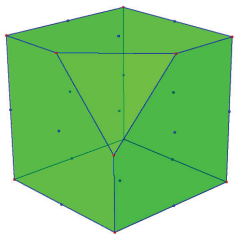

Figure 10 A Fano polytope, which is not reflexive

From now on, we will focus on the special properties of reflexive polytopes.

How many equivalence classes of reflexive polytopes are there in dimension n? In one dimension there is only one reflexive polytope, namely, the closed interval $[-1, 1]$. (It follows that the one-dimensional reflexive polytope is its own polar polytope.) We have seen the 16 equivalence classes of two-dimensional reflexive polygons. The physicists, Maximilian Kreuzer and Harald Skarke, counted equivalence classes of reflexive polytopes in dimensions three and four. Their results are summarized in TABLE 1; a description of a representative polytope from each class may be found at [**6**].

TABLE 1: Counting Reflexive Polytopes

Dimension	**Classes of Reflexive Polytopes**
1	1
2	16
3	4,319
4	473,800,776
≥ 5	??

The physicists' method for classifying polytopes was very computationally intensive, so it is not effective in higher dimensions. In dimensions five and higher, the number of equivalence classes of reflexive polytopes is an open problem.

The connection to string theory

String theory and mirror families Why were physicists classifying reflexive polytopes? As we noted in the introduction, the answer lies in a surprising prediction made by string theory.

String theory is one candidate for what physicists call a *Grand Unified Theory*, or GUT for short. A Grand Unified Theory would unite the theory of general relativity with the theory of quantum physics. General relativity is an effective description for the way our universe works on a very large scale, at the level of stars, galaxies, and black holes. The theory of quantum physics, on the other hand, describes the way our universe works on a very small scale, at the level of electrons, quarks, and neutrinos. Attempts to combine the theories have failed: standard methods for "quantizing" physical theories don't work when applied to general relativity, because they predict that empty space should hold infinite energy.

String theory solves the infinite energy problem by re-defining what a fundamental particle should look like. We often imagine electrons as point particles, that is, zero-dimensional objects. According to string theory, we should treat the smallest components of our universe as one-dimensional objects called *strings*. Strings can be open, with two endpoints, or they can be closed loops. They can also vibrate with different amounts of energy. The different vibration frequencies produce all the particles that particle physicists observe: quarks, electrons, photons, and so forth.

We are accustomed to thinking of point particles as located somewhere in four dimensions of space and time. But string theory insists on something more. To be consistent, string theory requires that strings extend beyond the familiar four dimensions, into extra dimensions. The extra dimensions must have particular geometric shapes. Mathematically, these shapes are known as *Calabi-Yau manifolds*.

To construct a string-theoretic model of the universe, we must choose a particular Calabi-Yau manifold to represent the extra dimensions at a point in four-dimensional space-time. Further, there are multiple ways to use this manifold. We will consider two of these theories, the *A-model* and the *B-model*. These models have very similar definitions: The only difference is whether one works with a particular variable or its complex conjugate. However, the physical consequences predicted by the A-model and the B-model for a particular geometric space are quite different.

When physicists began to explore the implications of string theory, they stumbled on a surprising correspondence: Different sets of choices can yield the same observable physics. As physicists studied the A-model and the B-model for many Calabi-Yau manifolds, they discovered pairs of Calabi-Yau manifolds where the A-model for the first manifold in the pair made the same predictions as the B-model for the second manifold, and vice versa. They hypothesized that a *mirror manifold* should exist for every Calabi-Yau manifold (or at least every Calabi-Yau manifold where the A-model and B-model can be defined).

In more mathematical terms, the physicists' hypothesis implies that Calabi-Yau manifolds should arise in paired or *mirror* families. (Mathematicians prefer to work with multi-parameter families of Calabi-Yau manifolds, rather than individual manifolds, because moving from one point of four-dimensional space-time to a nearby point might deform the shape of the attached Calabi-Yau manifold.) Since the geometric properties of each family determine the same physical theories, we can use information about the geometry of one family to study the properties of the mirror spaces.

What does this have to do with reflexive polytopes?

We can use reflexive polytopes to describe mirror families! To do so, we need a recipe that starts with a reflexive polytope and that produces a geometric space. We will proceed in two steps: First, we will use our polytope to build a family of polynomials, and then we will use our polynomials to describe a family of geometric spaces. (Technically, we will work with Laurent polynomials, which can involve negative powers.) We will obtain a mirror pair of families of geometric spaces corresponding to each mirror pair of polytopes:

$$\begin{array}{ccc} \text{polytope} & \longleftrightarrow & \text{polar polytope} \\ \downarrow & & \downarrow \\ \text{Laurent polynomial} & \longleftrightarrow & \text{mirror Laurent polynomial} \\ \downarrow & & \downarrow \\ \text{family of spaces} & \longleftrightarrow & \text{mirror family} \end{array}$$

From polytopes to polynomials Let Δ be a reflexive polytope. We want to construct a family of polynomials using Δ. We start by defining the variables for our polynomial. We do so by associating the variable z_i to the ith standard basis vector in the lattice N:

$$\begin{aligned} (1, 0, \dots, 0) &\leftrightarrow z_1 \\ (0, 1, \dots, 0) &\leftrightarrow z_2 \\ &\vdots \\ (0, 0, \dots, 1) &\leftrightarrow z_n \end{aligned}$$

Think of the z_i as complex variables: We will let ourselves substitute any nonzero complex number for z_i.

Next, for each lattice point in the polar polytope Δ°, we define a monomial, using the following rule:

$$(m_1, \dots, m_k) \leftrightarrow z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k}.$$

Finally, we multiply each monomial by a complex parameter α_j, and add up the monomials. This gives us a family of polynomials parameterized by the α_j.

Let's work out what this step looks like in the case of the one-dimensional reflexive polytope $\Delta = [-1, 1]$. Because we are working with a one-dimensional lattice N, there is only one standard basis vector, namely 1. Corresponding to this basis vector, we have one monomial, z_1. Next we consider the polar polytope Δ°. The one-dimensional reflexive polytope is its own polar dual, so $\Delta^\circ = [-1, 1]$. Thus, Δ° has three lattice points, -1, 0, and 1. From each of these lattice points, we build a monomial, as follows:

$$\begin{aligned} -1 &\mapsto z_1^{-1} \\ 0 &\mapsto 1 \\ 1 &\mapsto z_1 \end{aligned}$$

Finally, we multiply each monomial by a complex parameter and add the results. We obtain the family of Laurent polynomials $\alpha_1 z_1^{-1} + \alpha_2 + \alpha_3 z_1$, which depends on the three parameters α_1, α_2, and α_3. Notice that, because z_1 is raised to a negative power in the first term, we cannot allow z_1 to be zero.

Next, let's look at the family of Laurent polynomials corresponding to the reflexive triangle in FIGURE 2(a). We are now working with a two-dimensional polytope, so we have two variables, z_1 and z_2. The polar polygon of our reflexive triangle is shown in FIGURE 6. It contains ten lattice points (including the origin), so we will have ten monomials.

$$\begin{aligned}(-1, 2) &\mapsto z_1^{-1}z_2^2\\ (-1, 1) &\mapsto z_1^{-1}z_2\\ (0, 1) &\mapsto z_2\\ &\vdots\\ (2, -1) &\mapsto z_1^2z_2^{-1}\end{aligned}$$

When we multiply each monomial by a complex parameter and add the results, we obtain a family of Laurent polynomials of the form

$$\alpha_1 z_1^{-1}z_2^2 + \alpha_2 z_1^{-1}z_2 + \alpha_3 z_2 + \alpha_4 z_1^{-1} + \alpha_5 + \alpha_6 z_1$$
$$+ \alpha_7 z_1^{-1}z_2^{-1} + \alpha_8 z_2^{-1} + \alpha_9 z_1 z_2^{-1} + \alpha_{10} z_1^2 z_2^{-1}.$$

The mirror family of polynomials is obtained from the big reflexive triangle in FIGURE 6. We are still working in two dimensions, so we still need two variables; let's call these w_1 and w_2. The big triangle's polar polygon is the triangle in FIGURE 2(a), since the polar of the polar dual of a polygon is the original polygon. Thus, the mirror family of polynomials will only have four terms, corresponding to the four lattice points of the triangle in FIGURE 2(a). It is given by

$$\beta_1 w_1^{-1}w_2^{-1} + \beta_2 w_2 + \beta_3 + \beta_4 w_1.$$

From polynomials to spaces If we set a Laurent polynomial equal to zero, the resulting solutions describe a geometric space. Let's look at some examples using the family $\alpha_1 z_1^{-1} + \alpha_2 + \alpha_3 z_1$ obtained from the one-dimensional reflexive polytope. If we set $\alpha_1 = -1$, $\alpha_2 = 0$, and $\alpha_3 = 1$, we obtain the polynomial $-z_1^{-1} + z_1 = 0$. Solving, we find that $z_1^2 = 1$, so the solutions are the pair of points 1 and -1. If we set $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 1$, we obtain the polynomial $z_1^{-1} + z_1 = 0$. In this case, we find that $z_1^2 = -1$, so the solutions are the pair of points i and $-i$. (Now we see why it is important to work over the complex numbers!)

As we vary the parameters α_1, α_2, and α_3, we will obtain all pairs of nonzero points in the complex plane. Since the one-dimensional reflexive polytope is its own polar dual, the mirror family will also correspond to pairs of nonzero points in the complex plane. These are zero-dimensional geometric spaces inside a one-complex-dimensional ambient space. To describe more interesting geometric spaces, we'll have to increase dimensions.

What are the spaces corresponding to the mirror pair of triangles in FIGURE 6? Let's set the Laurent polynomials corresponding to the smaller triangle equal to zero.

$$\alpha_1 z_1^{-1}z_2^2 + \alpha_2 z_1^{-1}z_2 + \alpha_3 z_2 + \alpha_4 z_1^{-1} + \alpha_5 + \alpha_6 z_1$$
$$+ \alpha_7 z_1^{-1}z_2^{-1} + \alpha_8 z_2^{-1} + \alpha_9 z_1 z_2^{-1} + \alpha_{10} z_1^2 z_2^{-1} = 0.$$

We can multiply through by $z_1 z_2$ without changing the nonzero solutions. We obtain

$$\alpha_1 z_2^3 + \alpha_2 z_2^2 + \alpha_3 z_1 z_2^2 + \alpha_4 z_2 + \alpha_5 z_1 z_2 + \alpha_6 z_1^2 z_2 + \alpha_7 + \alpha_8 z_1 + \alpha_9 z_1^2 + \alpha_{10} z_1^3 = 0.$$

Let's re-order, so that terms of higher degree come first:

$$\alpha_{10} z_1^3 + \alpha_1 z_2^3 + \alpha_6 z_1^2 z_2 + \alpha_3 z_1 z_2^2 + \alpha_9 z_1^2 + \alpha_2 z_2^2 + \alpha_5 z_1 z_2 + \alpha_8 z_1 + \alpha_4 z_2 + \alpha_7 = 0.$$

As we vary our parameters α_i, we will obtain all possible degree-three or *cubic* polynomials in two complex variables. We cannot graph the solutions to these polynomials, because they naturally live in two complex (or four real) dimensions. However, we can graph the solutions that happen to be pairs of real numbers. These will trace out a curve in the plane. The real solutions for two possible choices of the parameters α_i are shown in FIGURES 11 and 12.

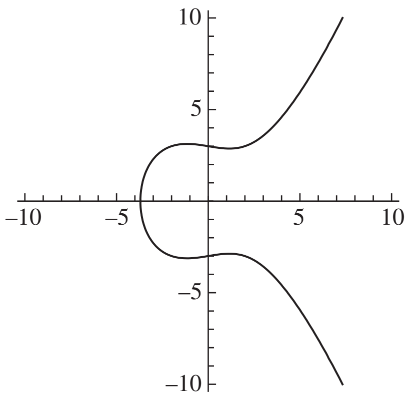

Figure 11 A real cubic curve

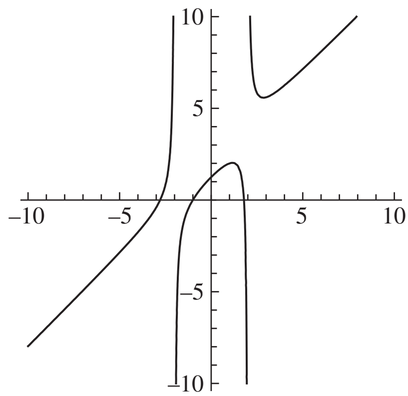

Figure 12 Another cubic curve

The mirror family of spaces is given by solutions to

$$\beta_1 w_1^{-1} w_2^{-1} + \beta_2 w_2 + \beta_3 + \beta_4 w_1 = 0.$$

We can multiply through by $w_1 w_2$ without changing the nonzero solutions:

$$\beta_1 + \beta_2 w_1 w_2^2 + \beta_3 w_1 w_2 + \beta_4 w_1^2 w_2 = 0.$$

Our mirror family of spaces also consists of solutions to cubic polynomials, but instead of taking all possible cubic polynomials, we have a special subfamily.

Physicists are particularly interested in *Calabi-Yau threefolds*: These three complex-dimensional (or six real-dimensional) spaces are candidates for the extra dimensions of the universe. We can generate Calabi-Yau threefolds using reflexive polytopes. For instance, one of the spaces in the family corresponding to the polytope with vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$ can be described by the polynomial

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 = 0.$$

Although we cannot graph this six-dimensional space, we can begin to understand its complexity by drawing a two-dimensional slice in $\mathbb{R}^3$. One possible slice is shown in FIGURE 13. You can generate and rotate slices of this space at the website [**4**].

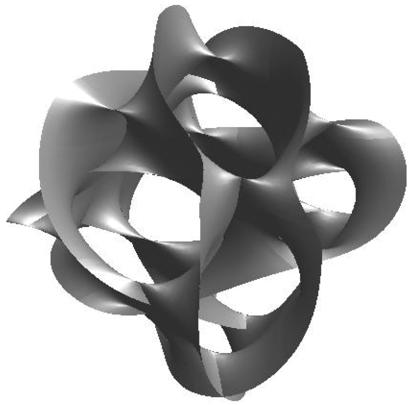

Figure 13 Slice of a Calabi-Yau threefold

Physical, combinatorial, and geometrical dualities String theory inspired physicists to study the geometric spaces known as Calabi-Yau manifolds. Using the duality between pairs of reflexive polytopes, we have written a recipe for constructing mirror families of these manifolds:

$$\begin{array}{ccc} \text{polytope} & \longleftrightarrow & \text{polar polytope} \\ \downarrow & & \downarrow \\ \text{Laurent polynomials} & \longleftrightarrow & \text{mirror Laurent polynomials} \\ \downarrow & & \downarrow \\ \text{spaces} & \longleftrightarrow & \text{mirror spaces} \end{array}$$

The ingredients in our recipe are combinatorial data, such as the number of points in a lattice polytope; the results of our recipe are paired geometric spaces. Combinatorics not only allows us to cook up these spaces, it gives us a way to study them: We can investigate geometric and topological properties of Calabi-Yau manifolds by measuring the properties of the polytopes we started with.

The geometric properties of a Calabi-Yau manifold V are encoded in a list of nonnegative integers known as *Hodge numbers*. (Readers who have studied algebraic topology will recognize the vector spaces $H^n(V, \mathbb{C})$, whose dimensions are given by the *Betti numbers*; the Hodge numbers tell us how to break up these vector spaces into smaller subspaces, using results from complex analysis.)

Two of the Hodge numbers, $a(V)$ and $b(V)$, count ways in which V can be deformed. In the physicists' language, the Hodge number $a(V)$ counts the number of *A-model variations*; mathematically, these are the number of independent ways to deform the notion of distance, or *Kähler metric*, on V. The Hodge number $b(V)$ counts the number of *B-model variations*, that is, the ways to deform the *complex structure* of V. (The complex structure tells us how to find local coordinate patches for V that look like subspaces of $\mathbb{C}^k$.)

Before physicists arrived on the scene, these two types of deformations were the provinces of two different fields of mathematics: Differential geometers studied distance and metrics, using the tools of differential equations, while algebraic geometers studied complex structures, relying on the power of modern algebra. Mirror symmetry predicts that, for Calabi-Yau manifolds, these deformations are intimately related. If V and V° are a mirror pair of Calabi-Yau manifolds, then their possible A-model variations and B-model variations must be reversed. Physicists conjectured that, given a Calabi-Yau manifold V, we should be able to find a mirror manifold V° with the appropriate Hodge numbers:

$$a(V) = b(V^\circ) \quad \text{and} \quad b(V) = a(V^\circ).$$

When physicists first framed this conjecture, very few examples of Calabi-Yau manifolds were known. Reflexive polytopes provide both a rich source of example Calabi-Yau manifolds, and a concrete mathematical construction of their mirrors. In the early 1990s, Victor Batyrev discovered and proved formulas for $a(V)$ and $b(V)$, which work when V is obtained from a reflexive polytope Δ of dimension $k \geq 4$:

$$a(V) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$b(V) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ).$$

Here $\ell()$ is the number of lattice points in a polytope or face, and $\ell^*()$ is the number of lattice points in the interior of a polytope or face. (For a face, this means that $\ell^*()$ does not count lattice points on its boundary.) The Γ are codimension 1 faces of Δ (that is, faces of dimension $k-1$), Θ are codimension 2 faces of Θ (that is, faces of dimension $k-2$), and $\hat{\Theta}$ is the face of Δ° dual to Θ; similarly, Γ° are codimension 1 faces of Δ°, Θ° are codimension 2 faces of Θ°, and $\hat{\Theta}^\circ$ is the face of Δ dual to Θ°. Notice that the variations of complex structure are controlled by the number of lattice points in the polar polytope Δ°. This is reasonable, because each lattice point in Δ° corresponds to a monomial in the equation for V, and in turn the equation for V determines a complex structure.

Since $(\Delta^\circ)^\circ = \Delta$, it follows immediately that the Calabi-Yau manifolds V° obtained from Δ° satisfy

$$a(V) \;=\; b(V^\circ) \text{ and } b(V) \;=\; a(V^\circ)\,.$$

Thus, Batyrev was able to use reflexive polytopes to turn a conjecture motivated by physics into a solid mathematical theorem.

In this article, we constructed Calabi-Yau manifolds as $(k-1)$-dimensional spaces described by a single equation in k complex variables. We can generalize this construction to build $(k-r)$-dimensional Calabi-Yau manifolds described by a system of r equations in k variables. The starting point is a reflexive polytope Δ. To define the r equations, we must divide the vertices of Δ into r disjoint subsets. The subsets of vertices define polytopes of lower dimension, and we can guarantee that the corresponding equations describe a Calabi-Yau manifold by placing conditions on these polytopes. Lev Borisov introduced a method for constructing a mirror Calabi-Yau manifold, corresponding to r more lower-dimensional polytopes. Together, Batyrev and Borisov proved a very general form of the equality

$$a(V) = b(V^\circ) \quad \text{and} \quad b(V) = a(V^\circ).$$

Their proof was indirect: They used the combinatorial data of the various polytopes to define a polynomial, and then showed that the coefficients of this polynomial correspond to Hodge numbers. (See [**1**] for an overview of this material.) Very recently, in

[2], the first author and a collaborator found closed-form expressions for a and b for Calabi-Yau threefolds described by two equations; these formulas directly generalize Batyrev's lattice point counting formulas from the single-equation case.

Reflexive polytopes remain the most bountiful source for examples of Calabi-Yau manifolds, and the simple combinatorial duality between polar polytopes continues to provide insight into geometrical dualities (mirror spaces) and even physical dualities (mirror universes)!

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FURTHER READING

1. David A. Cox and Sheldon Katz, *Mirror Symmetry and Algebraic Geometry*, American Mathematical Society, Providence, RI, 1991. The standard graduate-level text on mathematical mirror symmetry.
2. Charles F. Doran and Andrey Novoseltsev, Closed form expressions for Hodge numbers of complete intersection Calabi-Yau threefolds in toric varieties, *Contemporary Mathematics* **527** (2010) 1–14. http://dx.doi.org/10.1090/conm/527/10398. Explicit equations for Hodge numbers of Calabi-Yau threefolds in terms of polytope data.
3. Brian Greene, *The Elegant Universe*, Vintage Books, New York, 2003. A popular introduction to string theory by one of the field's pioneers. Chapter 10 describes physicists' discovery of mirror symmetry.
4. Andrew J. Hanson and Jeff Bryant, Calabi-Yau space, Wolfram Demonstrations Project, http://demonstrations.wolfram.com/CalabiYauSpace/. Mathematica code for visualizing a Calabi-Yau threefold.
5. Sheldon Katz, *Enumerative Geometry and String Theory*, American Mathematical Society, Providence, RI, 2006. This book, based on lectures for advanced undergraduate students at the Park City Mathematics Institute, describes another surprising connection between string theory and mathematics: We can use ideas from string theory to count subspaces of geometric spaces, such as the number of curves on a Calabi-Yau threefold.
6. Maximilian Kreuzer and Harald Skarke, Calabi-Yau data, http://hep.itp.tuwien.ac.at/~kreuzer/CY/. Data on the classification of reflexive polytopes.
7. Benjamin Nill, Gorenstein toric Fano varieties, preprint, http://arxiv.org/abs/math/0405448 (May 24, 2004). A discussion of Fano and reflexive polytopes, and their relationship to geometric spaces called toric varieties. A version of this paper was published in *Manuscripta Mathematica* **116** (2005) 183–210, but only the preprint gives the classification of reflexive polygons.
8. Bjorn Poonen and Fernando Rodriguez-Villegas, Lattice polygons and the number 12, *Amer. Math. Monthly* **107** (2000) 238–250. http://dx.doi.org/10.2307/2589316. Why is the sum of the lattice points on the boundary of a mirror pair of polygons always 12? Proofs involving combinatorics and number theory.
9. Shing-Tung Yau and Steve Nadis, *The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions*, Basic Books, New York, 2010. Yau won the Fields Medal for his work on Calabi-Yau manifolds. He teamed up with a science journalist to write this book, aimed at a general audience, on the geometry of Calabi-Yau manifolds and the implications for string theory.

Summary We describe special kinds of polygons, called Fano polygons or reflexive polygons, and their higher-dimensional generalizations, called reflexive polytopes. Pairs of reflexive polytopes are related by an operation called polar duality. This combinatorial relationship has a deep and surprising connection to string theory: One may use reflexive polytopes to construct "mirror" pairs of geometric spaces called Calabi-Yau manifolds that could represent extra dimensions of the universe. Reflexive polytopes remain a rich source of examples and conjectures in mirror symmetry.

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