BMC Intermediate II

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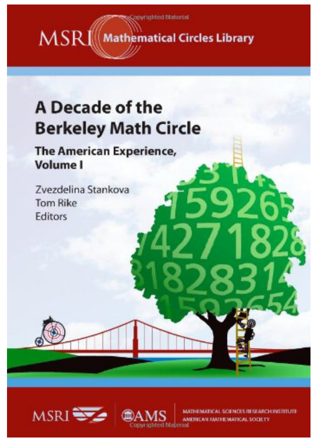
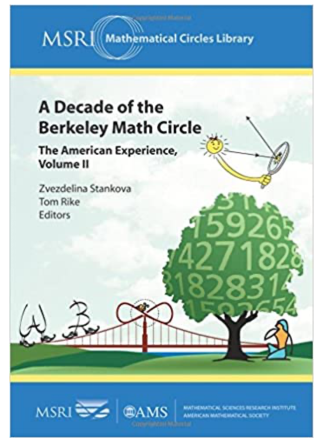
Induction or Not? Part II

with Zvezdelina Stankova

BMC-Upper Director





⓪ To be or not to be MMI?

(a) (Euler) $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}$

(b) (Riemann) All non-trivial zeros of $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$ have real part $=\frac{1}{2}$.

① Summations by MMI or not?

(a) $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$

(b) $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

(c) $1^3+2^3+3^3+\cdots+n^3=\left(\dfrac{n(n+1)}{2}\right)^2$

(d) $1+3+5+\cdots+2n-1=n^2$

(e) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{(n-1)\cdot n}=1-\dfrac{1}{n}$

(f) $\left(1-\dfrac{1}{4}\right)\left(1-\dfrac{1}{9}\right)\left(1-\dfrac{1}{16}\right)\cdots\left(1-\dfrac{1}{n^2}\right)=\dfrac{n+1}{2n}$

$$=\frac{2^2-1}{2^2}\cdot\frac{3^2-1}{3^2}\cdot\frac{4^2-1}{4^2}\cdot\frac{5^2-1}{5^2}\cdots\frac{(n-1)^2-1}{(n-1)^2}\cdot\frac{n^2-1}{n^2}\cdots$$

(g) $1+2+2^2+2^3+\cdots+2^n=?$

(h) $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\cdots+n\cdot 2^{n-1}=?$

② Prove $1^3+2^3+3^3+\cdots+n^3 \overset{?}{=} \left(\frac{n(n+1)}{2}\right)^2 \longrightarrow S_n,\ n\geq 1$

(a) By induction. (b) Algebraically. (c) Geometrically.

(B.S.) $n=1$: $1^3=1 \overset{?}{=} \left(\frac{1\cdot 2}{2}\right)^2 = 1^2$ ✓

(I.H.) Assume S_n is true for some $n\geq 1$.

(I.S.) To show S_{n+1} is also true.

S_{n+1}: $\underbrace{1^3+2^3+3^3+\cdots+n^3}_{S_n}+(n+1)^3 \overset{?}{=} \left(\frac{(n+1)(n+2)}{2}\right)^2$ ✓

$$\text{LHS} \overset{IH}{=} \left(\frac{n(n+1)}{2}\right)^2+(n+1)^3 = \frac{(n+1)^2\left[n^2+(n+1)\cdot 4\right]}{4}$$

$$= \frac{(n+1)^2\left[n^2+4n+4\right]}{4} = \frac{(n+1)^2\cdot(n+2)^2}{4}$$

③ Prove $\underbrace{1^3+2^3+3^3+\cdots+n^3}_{S_3}=\left(\frac{n(n+1)}{2}\right)^2$

(b) Algebraically.

$$S_0=1^0+2^0+3^0+\cdots+n^0=n$$

$$S_1=1^1+2^1+3^1+\cdots+n^1=\frac{n(n+1)}{2}$$

$$S_2=1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$S_3=1^3+2^3+3^3+\cdots+n^3\overset{?}{=}\left(\frac{n(n+1)}{2}\right)^2$$

$$S_4=1^4+2^4+3^4+\cdots+n^4\overset{?}{=}\text{Who knows?}$$

$$\begin{array}{c|lllll}
x & (x+1)^4= x^4 & +4x^3 & +6x^2 & +4x^1 & +1 \\
\hline
0 & \cancel{1^4}=0^4 & +4\cdot 0^3 & +6\cdot 0^2 & +4\cdot 0^1 & +1 \\
1 & \cancel{2^4}=\cancel{1^4} & +4\cdot 1^3 & +6\cdot 1^2 & +4\cdot 1^1 & +1 \\
2 & \cancel{3^4}=\cancel{2^4} & +4\cdot 2^3 & +6\cdot 2^2 & +4\cdot 2^1 & +1 \\
3 & \cancel{4^4}=\cancel{3^4} & +4\cdot 3^3 & +6\cdot 3^2 & +4\cdot 3^1 & +1 \\
\vdots & & & & & \\
n-2 & \cancel{(n-1)^4}=\cancel{(n-2)^4} & +4\cdot(n-2)^3 & +6\cdot(n-2)^2 & +4\cdot(n-2)^1 & +1 \\
n-1 & \cancel{n^4}=\cancel{(n-1)^4} & +4\cdot(n-1)^3 & +6\cdot(n-1)^2 & +4\cdot(n-1)^1 & +1 \\
n & (n+1)^4=\cancel{n^4} & +4\cdot n^3 & +6\cdot n^2 & +4\cdot n & +1 \\
\hline
\end{array}$$

$$(n+1)^4=4\cdot S_3+6\cdot S_2+4\cdot S_1+S_0+1$$

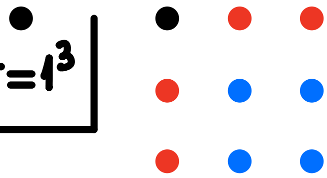
$$\Longrightarrow 4S_3 = (n+1)^4 - \cancel{6}\cdot\frac{n(n+1)(2n+1)}{\cancel{6}} - \cancel{4}^{\,2}\cdot\frac{n(n+1)}{\cancel{2}} - n - 1$$

$$4S_3 = (n+1)\left((n+1)^3 - 2n^2 - n - 2n - 1\right)$$
$$4S_3 = (n+1)\left((n+1)^3 - 2n(n+1) - (n+1)\right)$$
$$4S_3 = (n+1)\left((n+1)^3 - (n+1)(2n+1)\right)$$
$$4S_3 = (n+1)(n+1)\left((n+1)^2 - (2n+1)\right)$$
$$4S_3 = (n+1)^2\left(n^2 + \cancel{2n+1} - \cancel{2n-1}\right)$$

$$\Longrightarrow \boxed{S_3 = \frac{(n+1)^2 n^2}{4} = \left(\frac{(n+1)n}{2}\right)^2}$$

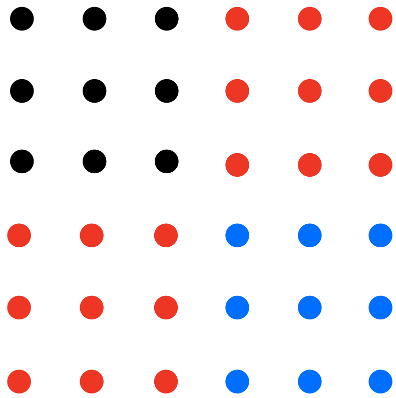
④ Prove $1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2=(1+2+3+\cdots+n)^2$

(c) Geometrically.

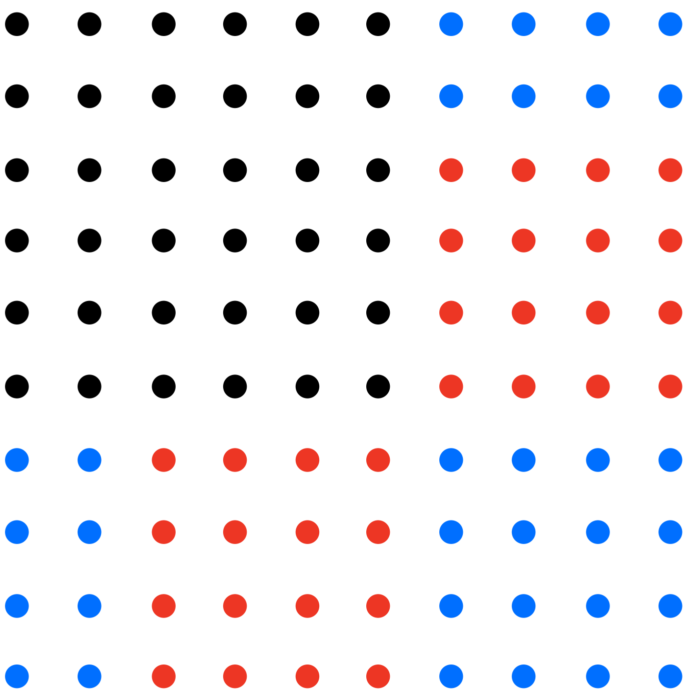


$1^2=1^3$

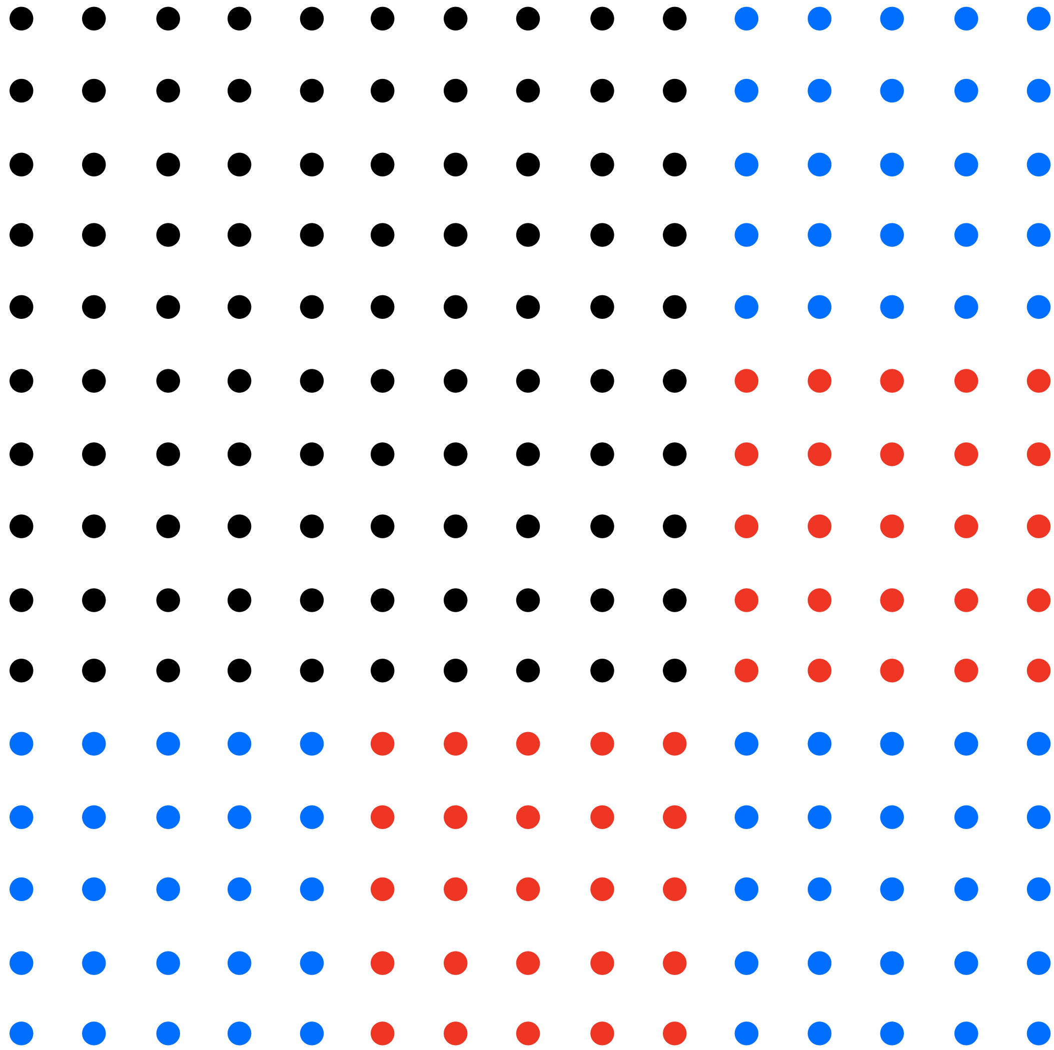
3^2



6^2



10^2



In general:

$$\overbrace{\left(\frac{(n-1)\cdot n}{2}\right)^2}^{\text{previous}} + n\cdot \overbrace{(1+2+\cdots+(n-1))\cdot 2}^{\text{left-over}} + n^2$$

$$\begin{array}{rl} & 1 + 2 + \cdots + (n-1) \\ + & n-1 + n-2 + \cdots + 1 \\ \hline & n + n + \cdots + n \\ & = n\cdot(n-1) \end{array}$$

$$n\cdot n(n-1) + n^2$$
$$= n^2(n-1+1) = n^2\cdot n = n^3$$

⑤ Creating Change

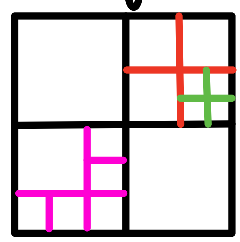
- \$3, \$5 bills
- What sums can we pay?

n	1	2	3	4	5	6	7	8	9	10	11	12	13
Y/N	X	X	✓	X	✓	✓	X	✓	✓	✓	✓	✓	✓

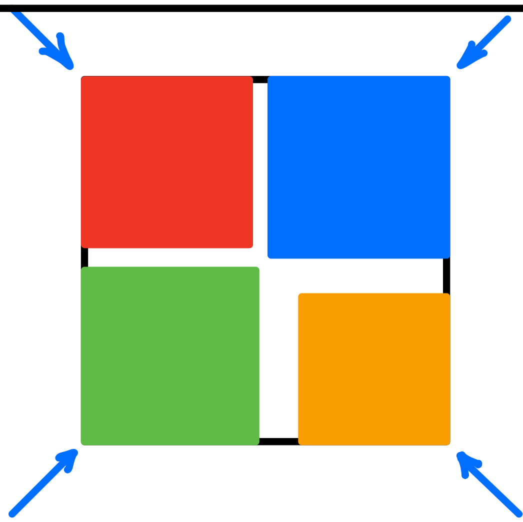
⑥ Squares

Into how many squares can we cut a square?

n	1	2	3	4	5	6	7	8	9	10	11	12	13
Y/N	✓	X	X	✓	X	✓	✓	✓	✓	✓	✓	✓	✓



Prove: this is impossible for $n=2,3,5$.



By contradiction, assume it is possible to partition the big square into 5 smaller squares...

⑦ The Pie Fight

The Three Stooges
dailymotion.com

Laurel-and-Hardy.com

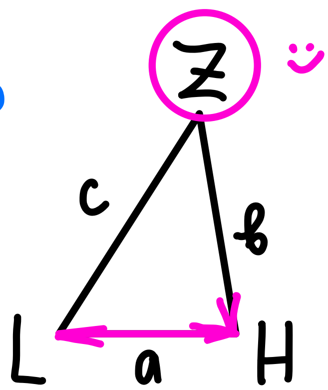
Given:
- odd # of people > 1
- all pairwise distances are different
- everyone pies the closest person

Prove: $\exists$ a survivor !

Proof: MMI

P_n: The statement is true for n odd, $n \geq 3$

(B.S.) $n = 3$
scalene Δ



WLOG, $a < b < c$

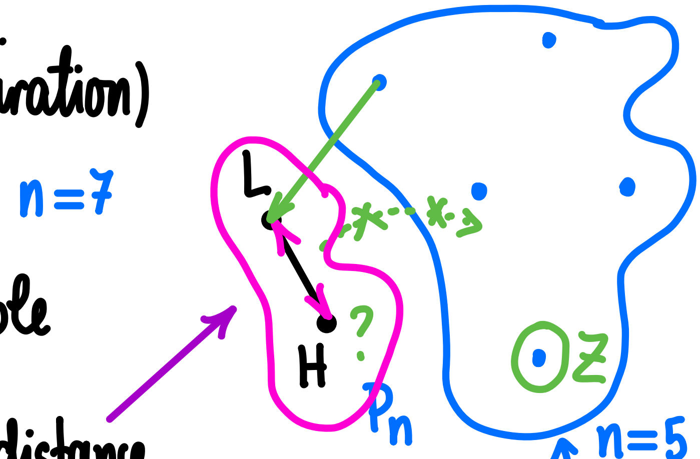
$\Rightarrow P_3$ is true. ✓

(I.H.) Assume P_n is true for some odd $n \geq 3$
⚠ Any valid configuration of $\underline{n}$ people.

(I.S.) To show P_{n+2} is true.
(for any valid configuration)

Key idea:
Use the Extreme Principle

longest distance and find survivor ☹ too much!	shortest distance and some who got pied for sure!



Play: another game!

Claim: Z survives the original game! too!

Pf (of Claim):
- L ⟷ H
- some pies could now be re-directed from new game to L or H
- No pies fly from {L, H} to anyone else.

$\Longrightarrow$ whoever survived the other game will survive the original game!

$\Longrightarrow$ Z survives!

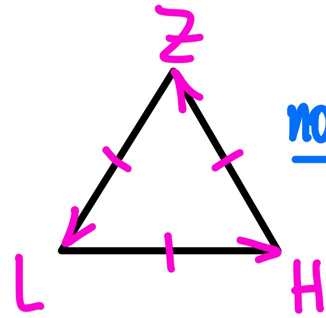
$\Longrightarrow$ P_{n+2} is also true.

Conclusion: All P_n's are true for odd $n \geq 3$

Burning Q_1:

Do all distances have to be different ?

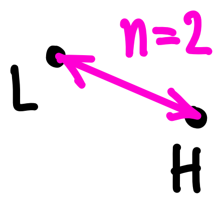
Counterexample



no survivor

Odd Q_2: Why is n odd ?

Counterexample



Hope: Perhaps true for larger even n's ?

$n = \underline{18}$

- marry off
- send off to honeymoons on different planets!