Recall the following two definitions in a graph G:

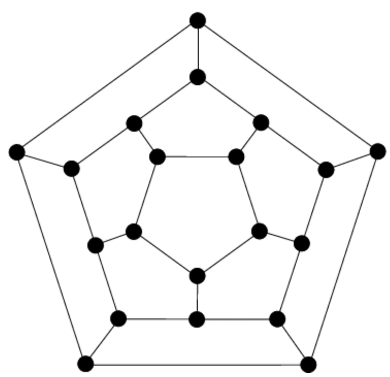
- A path/cycle is called Eulerian if it traverses every **edge** of G exactly once.
- A path/cycle is called Hamiltonian if it visits every **vertex** of G exactly once.

We saw that G admits an Eulerian cycle if and only if all of its vertices have even degree (and it is connected). Similarly, the graph G admits an Eulerian path if and only if it has ≤ 2 vertices of odd degree. We proved this 2^{nd} fact from the 1^{st} by adding in a "phantom edge" between two vertices of odd degree, and then finding a cycle, which was cut into just a path when we removed the "phantom edge."

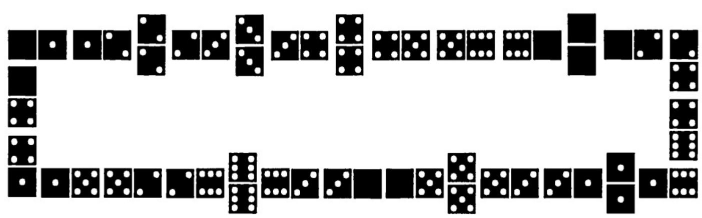
For each of the questions below, ask yourself the following:

- Is there a graph hidden in this framework?
- What are the vertices? What are the edges?
- What aspect of this graph do I want? Does the question deal with Eulerian or Hamiltonian paths/cycles, cycles in general, degrees, etc.?

1. You want to make a line drawing without tracing over any part twice. Given a design that you wish to draw in this way, how do you determine the fewest number of times that you must pick up your pen? What is this number for the "dodecahedral graph" shown below?



2. In the game of dominos, we are given some tiles, each labelled with two numbers between 0 and n. The tiles need to be laid down in such a way that any two adjacent numbers are equal. For example, we can arrange the full set of dominos for $n = 6$ in a cyclical fashion, as shown here:



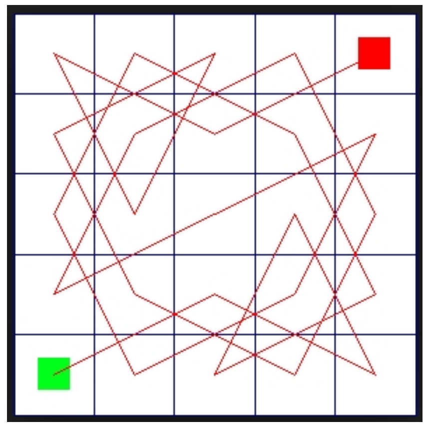
For which n is it possible to do this? More generally, if I hand you any collection of dominos, how can you determine whether it is possible to arrange them cyclically in this fashion?

3. In chess, recall that a knight moves in an L-shape (from one corner of a 2×3 rectangle to the opposite). A knight's tour of an $n \times n$ chessboard is a sequence of moves so that the knight visits each square exactly once. The tour is said to be closed if the knight can then return to the square where it began.

   (a) Prove that there is no knight's tour on a 3×3 chessboard. What about a 4×4 chessboard?

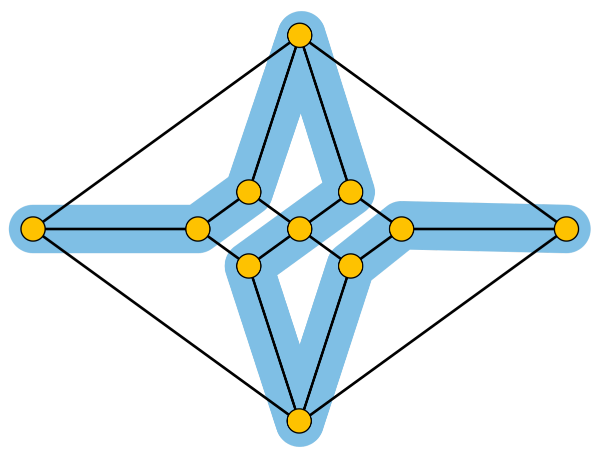
   (b) Prove that a bipartite graph with an odd number of vertices cannot admit a Hamiltonian cycle.

   (c) It is possible to find a knight's tour on a 5×5 chessboard, as shown here:

   However, use (b) to prove that there is no closed knight's tour on a 5×5.

   (d) Although the following graph has a Hamiltonian path, prove that it has no Hamiltonian cycle.

4. (a) Given a 6×6 grid, can you fill in exactly 14 squares, so that each row and each column contains an even number of filled squares? Can you do the same so that each row and each column contains an odd number of filled squares?

   (b) Given a 4×4 grid, can you fill in exactly 11 squares, so that each row and each column contains an odd number of filled squares? What about a 4×5 grid? What about a 5×5 grid?

   (c) Explore further! On an $m \times n$ grid, when is it possible to fill in q squares, so that each row and each column contains an even/odd number of filled squares? Can you find some general strategies to fill in squares? Can you prove some rules that show some cases to be impossible?

5. There is no "easy" strategy for determining whether an arbitrary graph admits a Hamiltonian path/cycle, but in some very nice scenarios, we can determine existence from simple conditions. In this problem, you will fill in steps to reprove the following theorem that we discussed near the end of our session:

   Ore's Theorem. Let G be a simple graph with n vertices, where any two distinct vertices v and w are either connected by an edge or satisfy $\deg(v) + \deg(w) \geq n$. Then G admits a Hamiltonian cycle.

   Proof. Assume that there is some graph G not satisfying the theorem (we call this a counterexample). We will derive a contradiction from this. First, argue that that:

   (a) We can choose G to have a maximal number of edges among all counterexamples on n vertices. (Hint: Is there a maximal number of edges that a simple graph on n vertices can possess?) Thus, we have a graph G on n vertices, such that:

      i. Any distinct vertices v and w are either connected by an edge or satisfy $\deg(v) + \deg(w) \geq n$;

      ii. G does not admit a Hamiltonian cycle (in particular, this G is *not* the complete graph K_n);

      iii. If we take any two vertices v and w that are not connected by an edge, and we add in a new edge connecting them, the resulting graph will admit a Hamiltonian cycle. (Use maximality.)

   (b) This graph G admits a Hamiltonian path. (Hint: Take two vertices not connected by an edge. Connect them by a "phantom edge" and apply (a.iii) to get some Hamiltonian cycle, which must involve this edge. If you then remove this edge, why do you still get a Hamiltonian path?)

   Say that our Hamiltonian path traverses the vertices $v_1, v_2, \ldots, v_n$, in that order. Then, show that:

   (c) The vertices v_1 and v_n are not connected by an edge and thus $\deg(v_1) + \deg(v_n) \geq n$.

   (d) There is some $1 < i \leq n$, such that $v_1 \sim v_i$ and $v_{i-1} \sim v_n$ (where $v \sim w$ means that the vertices v and w are connected by an edge). This will require a careful application of (c).

   (e) Show that G admits a Hamiltonian cycle. $\big($Hint: draw a picture of what we found in part (d).$\big)$

   We have now reached a contradiction, since we have proven that our particular "counterexample" G actually *does* admit a Hamiltonian cycle. Thus no counterexamples can exist, proving the theeorem. □

   (f) Now use Ore's theorem to prove Dirac's theorem, which is stated below.

   Dirac's Theorem. If G is a simple graph with n vertices, where each vertex v has $\deg(v) \geq n/2$, then G admits a Hamiltonian cycle.