BMC Spring 2021
Beginners

Math Behind Transformers, Part II

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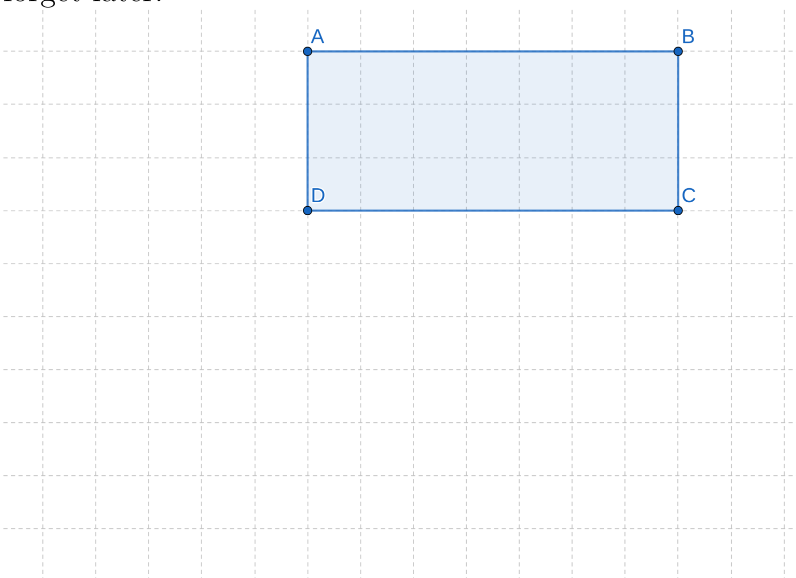
(Continued from last time)

Let's closely look at our famous technique of decomposing an arbitrary rectangle into a square: Perform that technique below. Write and explain the steps on the right so you don't forget later.



The key to this decomposition is the construction of the geometric mean of two numbers. In fact, this shows that we can construct the square root of a lot of numbers! Let's explore...

I. The Square Root and Its Constructibility

Definition I.1. Given a positive real number a, we define the square root of a, denoted by $\sqrt{a}$, to be the positive number such that $\sqrt{a} \times \sqrt{a} = a$.

Examples and Exercises:

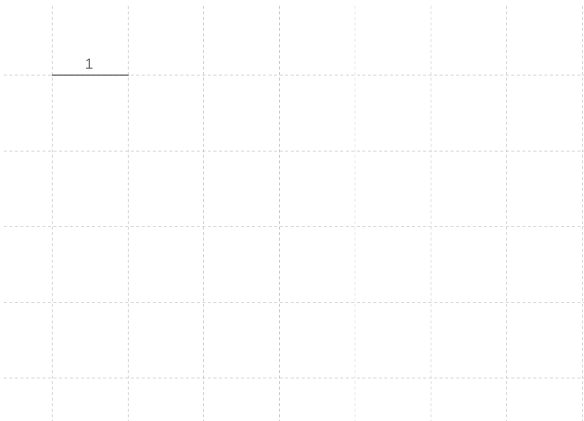
1. Find $\sqrt{1}$, $\sqrt{4}$, $\sqrt{13225}$.

2. True or False: Because $(-3) \times (-3) = 9$, we say that $\sqrt{9} = -3$.

3. True or False: For *any* x, $\sqrt{x^2} = x$
4. Side note: Are there fast ways to calculate the squares of certain 2-digit numbers?

Practical Applications Given a unit length below. Can you construct/draw the following lengths?

- $\sqrt{2}, \sqrt{3}$.
- $\sqrt{n}$, for any $n \in \mathbb{N}$.
- What about $\sqrt{\sqrt{2}}$?



I think we are ready to formulate the first big lemma! :)

Lemma I.1. If a positive length a is constructible, then $\sqrt{a}$ is also constructible.

Proof. By construction, we construct the geometric mean of a and 1. □

Then, are there numbers we cannot construct? The answer is yes. When you do some more algebra, we can explain why such a number as $\sqrt[3]{2}$ is not constructible. Other famous inconstructible numbers include π, e. It took a long time before mathematicians figured out why this is the case! Constructions started in Ancient Greece, but only until a few thousand years later that we finally proved that we cannot construct those special numbers. In short, we will leave this discussion for the future (perhaps only a few years instead of a few thousand).

II. The Geometric Meaning of Averages

For more curious students only. Before we explore the geometric meaning of the infamous averages, we need to discuss a new notion:

Definition II.1. A *locus* is a collection of points that share the same property, usually with reference to some fixed object(s).

Examples: circles := locus of points (on the same plane) equidistant to a fixed point (the center).

1. Those who are familiar with the Arithmetic Mean of two numbers (generally referred to as "the average") might remember that it represents the midpoint of that segment. We can explore this concept in the sense of invariances (properties that don't change).
   a) On the real line, mark $A = (3)$ and $B = (7)$, where is their average (A.M.)?
   b) What is the locus of points with the property that the sum of its distance to A and B is 4?
   c) What is special about their arithmetic mean in this locus?

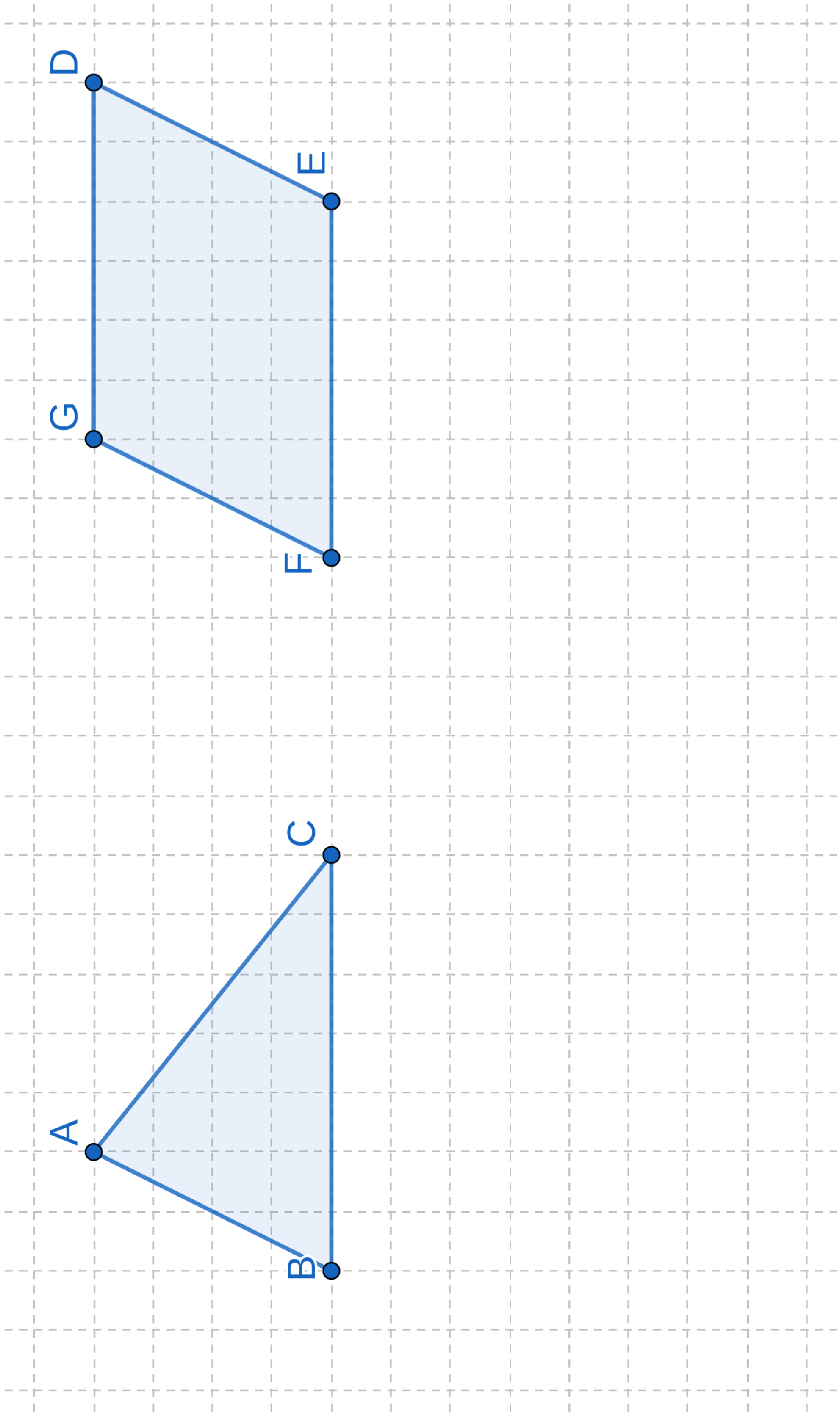
2. Is there an analogue in Geometric Mean? We know that the Geometric Mean has to do with area and rectangles, so let's start with a rectangle $ABCD$ of area 4 with a fixed vertex A at the origin (click here to see illustration).
   a) If we want to fix the area at 4, and only allow B to move on the x-axis, D to move on the y-axis, what is the locus of C as the rectangle varies? This will require some advanced algebra! We can experiment!
   b) What is special about the G.M. of AB and CD?

III. Squaring Problem (cont.)

Now, to return to our original question: Can we cut a certain shape and put it back into a square? Let's try a slightly harder shape:

> How do we cut the following triangle and parallelogram and turn them into squares?

(*Hint*: You might need to know how to calculate their area.)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