Exponential Generating Functions

Espen Slettnes

Berkeley Math Circle

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Generating functions are formal power series whose coefficients come from a sequence; the generating function for a sequence $(a_n)_{n\in\mathbb{Z}}$ is $\sum\limits_{n\in\mathbb{Z}} a_n x^n$.

They are good at handling a lot of sequences because they enable manipulation of all terms simultaneously. The first few such sequences that I can think of are the powers of 2, the Fibonacci numbers, and the Catalan numbers, and there are many more.

They are inadequate for some sequences — the generating function for $a_n = n!$ only converges at $x = 0$.

Exponential generating functions are a variant. The egf for a sequence $(a_n)_{n\in\mathbb{Z}}$ is $\sum\limits_{n\in\mathbb{Z}} a_n \frac{x^n}{n!}$. This, of course, can manage sequences like $a_n = n!$. Today we will be talking about some uses of egfs, and will answer the following questions.

Example 1. What is the number of derangements of $[n]$?

Example 2. What is the average number of cycles in a permutation of $[n]$?

Example 3. What is the number of permutations of $[n]$ with only odd-length/only even-length cycles?

Example 4. What is the egf for alternating permutations (ones that alternate between increases and decreases)?

Example 5. What is the number of rooted labeled trees on n vertices? Rooted labeled binary trees?