Invariants

Based on BMC book vol.1, p.206-219[1]

Date: 03/31/2021

1. **(Number games)**

   (a) **(Parities)** Generalize the exercise discussed in session as follows.

   (a) Start with n 0's and m 1's. Cross out any two of the digits; if the digits crossed out are the same, write a new 0; otherwise, a 1.
   Which number will be left in the end? (*Hint:* Does your answer depend on m, n, or both?)

   (b) Start instead with six 0's, five 1's, and four 2's. Replace two numbers at a time as follows: two same numbers are replaced by 0 and two different numbers by 1. What do you see in the end? Why?

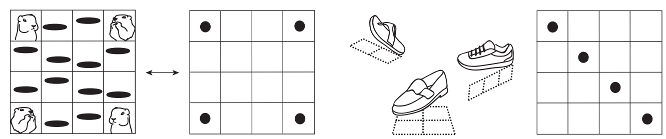
   (c) Generalize part (b) for n 0's, m 1's, and k 2's. Does the final outcome depend on how the game is played? Why?

   (b) **(Pointless Machine)** The pointless machine will be discussed in session if we have time.
   If initially only (7, 29) is available to the Pointless Machine, prove or disprove that the following pairs can be produced:

   (a) (29, 51); (b) (90,101); (c) (37, 92); (d) (39, 27); (e) (40, 84).

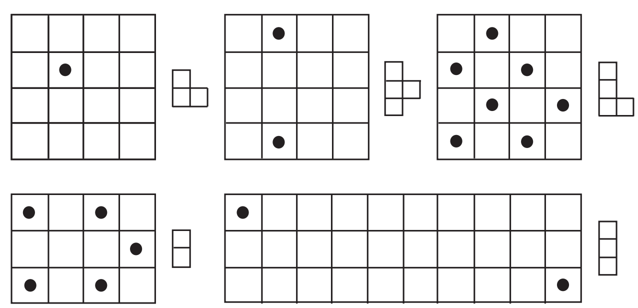
   (c) **(Extra problem, St. Petersburg '96)** Several positive integers are written on a blackboard. One can erase any two distinct numbers and write their gcd and lcm instead. Prove that eventually the numbers will stop changing.

2. **(Stomp)**



   (a) For each of the stomp board above and each of the three pieces above, find a way to clear the board in as few moves as possible, or prove that it cannot be done.

   (b) For each of the stomp board below, find a way to clear the board in as few moves as possible, or prove that it cannot be done. Use the stomp piece indicated on the right of each board.



[1]Worksheet adapted from that of Prof. Zvezdelina Stankova.