3.4. The Gopher Gun. Imagine now that we have a gun which can shoot along a whole row, column, or diagonal of the board, and each gopher in that row, column, or diagonal goes back down or comes up after the gun is shot. This is a cute interpretation of an original problem by Adam Hesterberg in my problem writing class at Mathcamp '05. It goes like this.

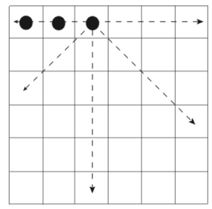

FIGURE 7. Gopher Gun

Problem 9. The squares of a 6×6 chessboard are each empty or contain a single dot (aka gopher). At each step you are allowed to reverse the state of all the squares in a single row, column, or diagonal (including the trivial diagonal consisting of a lone corner square). There are precisely three dots in the initial state of the board, located along the left-hand half of the top row (cf. Fig. 7). Prove that it is impossible to clear the board of dots.[5]

4. Tilings and More Invariants

No invariant session would be complete without *tilings*. Recall that to *tile* a figure with a particular shape means to cover that figure completely with copies of the given shape which do not overlap. For example, it is possible to tile a 4×4 board with eight copies of a 1×2 domino. In the following, we shall consider two shapes *the same* (or *congruent*) if they are obtained from one another by rotating, translating, flipping, or any combination of those.

Exercise 13 (Warm-up). Clearly, one can tile a 4×4 board with 2 copies of a 2×4 rectangle. It turns out that there are 5 other shapes, each built from eight 1×1 unit squares, such that 2 copies of each shape can also tile the 4×4 board. Find these other 5 shapes and tilings of the 4×4 board.

Exercise 14. There are 5 distinct shapes which can be constructed using four unit squares on a sheet of graph paper. These are called the *tetrominoes*. Draw all 5 tetrominoes and decide which ones can be used to tile a 4×4 board (one type of tetromino per tiling!).

Both Exercises 13 and 14 call for an intelligent trial and error approach. Obviously, symmetry (across the center of the board, or across a line) will play a part. You should organize your solutions in a systematic way so that they can generalize to larger boards of suitable sizes.

Problem 11. Show that it is impossible to cover a 4×5 rectangle with a complete set of tetrominoes, i.e., using each tetromino once.

Exercise 15 (Warm-up). Draw a rectangle 7 squares wide and 3 squares high, missing the middle square of the top row. Then find a way to tile this figure with one complete set of tetrominoes. Can you find another interesting shape consisting of 20 unit squares which can be built from one complete set of tetrominoes?

Problem 12. Consider a 7×7 board. Which square can be removed to allow a tiling with 3×1 trominoes of the resulting 48-square board? For example, removing a corner square is one answer: find a way to tile the resulting board. What are the remaining answers?

Our last tiling problem is probably the most interesting question here because it leads to some creative coloring schemes.

Problem 13. For each of the 5 tetrominoes, determine whether or not it can be used to tile a 6×6 board. In each case, find a way to do the tiling or demonstrate that it cannot be done using an appropriate coloring scheme.

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5. Escape of the Clones

This is a version of a famous puzzle attributed originally to Maksim Kontsevich, which appeared in the Tournament of the Towns and in the Russian journal *Kvant* in 1981 (cf. [**54, 50**]). Its solution will require the creation of invariants with *infinite series*.

5.1. The set-up of the game. Consider the first quadrant in the Cartesian plane divided into unit squares by horizontal and vertical lines at the positive integers. Place 3 dots (*clones*) in the shape of an L-tromino in the bottom left-most squares, and draw a "barbed wire fence" enclosing the dots and their 3 respective squares: this is the orange fence in Figure 8.

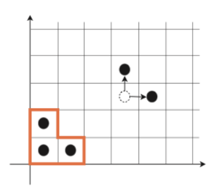

FIGURE 8. Escape of the Clones

5.2. The rules of the game. At each step you can erase a dot and replace it with two copies in adjacent squares, one directly above and the other directly to the right, as long as those squares are currently unoccupied. In other words, when a clone disappears, it sprouts two more clones above and to the right of it. Notice that this is a Stomp-like game

- whose board is the infinite first quadrant and
- whose "footprint" is an L-tromino allowed to be placed only in the standard orientation of the English letter L, and only when the corner square of L covers a clone while the other two squares of L land on clone-empty spots.

5.3. Freedom for the clones!

Problem 14 (Advanced). Prove that it is impossible to free all clones from the prison.

Although the problem setting is elementary enough for anyone to play and enjoy the game, the actual solution is hard to come up with and requires knowledge of a useful summation formula. That's great: while playing games, we will learn some algebra too!