Berkeley Math Circle

Sequences and Series

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March 31st and April 7th 2021

Warm-Up

Here are some lists of numbers. What patterns do you notice in each list? Can you write a rule to find the next number in the list? Can you write a rule to find any number in the list?

$a_n = 2, 4, 6, 8, \ldots$

$b_n = 99, 199, 299, \ldots$

$c_n = 3, -5, 7, -9, \ldots$

$d_n = 1, 4, 9, 16, \ldots$

$e_n = 2, 6, 18, \ldots$

$f_n = 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$

$g_n = 1, 8, 27, \ldots$

$h_n = 1, 1, 2, 3, 5, 8, \ldots$

PART I

1. Find the first 6 terms and 100th term of the sequence a_n for which

$$a_k = k^2 - 1$$

2. Find the first 6 terms and 100th term of the sequence defined recursively by the conditions

$$b_1 = 3, \qquad b_n = b_{n-1} + 2, \qquad for \;\; all \;\; n > 1$$

3. Find the 21st term of the sequence given by

$$a_n = 4n - 3$$

4. If n is a natural number, what is the 6th term of the sequence given by

$$a_n = 4(0.5)^n$$

5. Find the fifth term of the sequence

$$a_n = \left\{ \left(\frac{1}{n} \right)^{n-1} \right\}$$

6. Write the first 3 terms of the sequence. Determine whether the sequence converges or diverges. If it converges, give the limit.

   - $\left\{ \frac{1}{2^n} \right\}$
   - $\{2n - 3\}$
   - $\left\{ 2^n \right\}$
   - $\left\{ (0.1)^n \right\}$
   - $\left\{ \left(\frac{3}{2} \right)^n \right\}$

7. Write an explicit and recursive rule for the Arithmetic sequence.

   - 5, 8, 11, 14, ...
     Find a_{21}

   - 100, 94, 88, 82, ...
     Find a_{15}

8. Write an explicit and recursive rule for the Geometric sequence.

   - $2, \frac{3}{2}, \frac{9}{8}, \frac{27}{32}, \ldots$
     Find a_{11}

   - -0.3, 0.6, -1.2, 2.4, ...
     Find a_{20}

9. Write the first five terms of the sequence.

   - $a_1 = 3, \quad a_n = a_{n-1} - 7$

   - $b_1 = 4, \quad b_n = 2b_{n-1} - 1$

10. Write a recursive rule for the sequence.

    - 1, 1, 2, 3, 5, 8, ...

    - 1, 1, 2, 6, 24, ...

11. For the sequence defined by

$$a_n = n^2 - 5n + 2$$

what is the smallest value of n, given that n is a natural number, for which a_n is positive?

12. The nth term of a sequence is given by

$$a_n = 3n^2 - 1$$

Which term of the sequence is equal to 866?

13. The nth term of a sequence is given by

$$a_n = \frac{n^2 - 1}{n + 5}$$

Which term of the sequence is equal to 4? (Hint: $x^2 - a^2 = (x + a)(x - a)$).

PART II

Rewrite the following fractions as decimals.

$\frac{1}{10} =$	$\frac{1}{100} =$	$\frac{1}{9} =$	$\frac{1}{90} =$	$\frac{13}{99} =$
$\frac{1}{5} =$	$\frac{1}{50} =$	$\frac{2}{9} =$	$\frac{1}{45} =$	$\frac{257}{999} =$
$\frac{3}{10} =$	$\frac{3}{100} =$	$\frac{1}{3} =$	$\frac{1}{30} =$	$\frac{5}{11} =$.
$\frac{1}{2} =$	$\frac{1}{20} =$	$\frac{5}{9} =$	$\frac{5}{90} =$	$\frac{13}{990} =$
$\frac{3}{5} =$	$\frac{3}{50} =$	$\frac{2}{3} =$	$\frac{1}{15} =$	$\frac{7}{9000} =$
$\frac{7}{10} =$	$\frac{7}{100} =$	$\frac{7}{9} =$	$\frac{7}{90} =$	$\frac{17}{9900} =$

1. Describe a set of rules for determining a repeating decimal's pattern.

2. Based on the third column above rewrite the fraction $\frac{9}{9}$ as a decimal in two different ways.

3. What is the partial sum of the first seven triangular numbers?

4. What is the partial sum of the first eight natural numbers?

5. The corner of a stadium has 8 seats along the front row each successive row has two more seats that the row preceding it. How many seats are in the entire section?

6. Find the sum of the geometric sequence

$$\sum_{n=1}^{11} 4\left(\frac{1}{3}\right)^{n-1}$$

7. Write the sum of the finite arithmetic series in Sigma notation, then calculate the sum.

   - $2 + 6 + 10 + \cdots + 58$

   - $-1 + 4 + 9 + \cdots + 34$

8. Write the first 3 terms and the last term of the series.

- $$\sum_{i=1}^{22}(-9+11i)$$

- $$\sum_{i=4}^{22}(14+6i)$$

9. Write the first 3 terms and the last term of the geometric series, then calculate the sum.

- $$\sum_{i=1}^{22}4\left(\frac{1}{4}\right)^{i-1}$$

- $$\sum_{i=0}^{10}12\left(\frac{-1}{2}\right)^{i}$$

10. Find the sum, if it exists, using the formula for a converging infinite geometric series. If the series does not converge, state your reason.

- $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \ldots$

- $3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \ldots$

- $$\sum_{n=1}^{\infty} 5\left(\frac{3}{2}\right)^n$$

- $$\sum_{n=0}^{\infty} 5(0.75)^n$$

11. Calculate the indicated terms of the sequence defined by

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

12. Write the repeating decimal as a fraction in lowest terms.

    - 0.444...
    - 0.03232323232...
    - -1.268268268...
    - 5.27777777...

13. What fraction of the total area will be green?