BMC Advanced: Conformal mapping

211021 - Chris Overton (handout after day 1 of 2)

Suggested preparation for next week:
-- Review this handout, and bring any questions about it to class!
-- View the video (first reference at the end of this handout.)

A map is "conformal" if it preserves **angles** and **orientation**.
For that to even make sense, it must be "smooth": continuous and differentiable.
We focus on complex-differentiable ("**analytic**") maps $f : \mathbb{C} \to \mathbb{C}$, which are conformal (with one important exception!)

Plan for today:

- Quick intro to real differentiation
- Quick intro to complex numbers
- Why a complex derivative is so much more powerful than a real one
- $f(z) = z^2$ in greater depth
    - Is this conformal as a complex function?
- Stop discrimination: ∞ is a number too (well, sort of)
- Stereographic projection: a useful way to understand a complex map
- One very useful family of conformal maps: linear fractional "Möbius" transformations (as much as we have time for today)
- The Riemann mapping theorem, and how conformal maps can be useful

Next week:
-- More maps, more applications!

Quick intro/recap: real differentiation

If you look at a well-behaved curve under a magnifying glass, it looks more like a straight line.
The derivative of a straight line is just its slope (rise over run.)
So the derivative of a function is just this slope "in the limiting case":
$f'(x) = \frac{df}{dx} = lim_{h \to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Here, a finite change of x: $\Delta x = (x+h) - x$ ("the run") is written as h.
This results in a finite change of f: $\Delta f = f(x+h) - f(x)$, which is like a rise.
Then as Δx (=h) gets really small, $\frac{\Delta f}{\Delta x} \rightarrow \frac{df}{dx}$

The next examples show basic rules of differentiation:

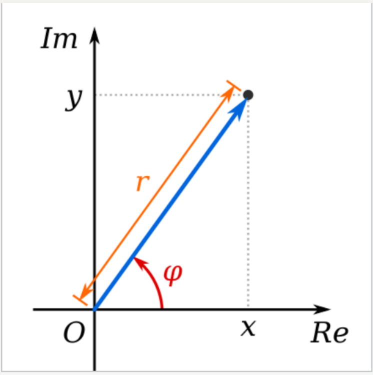
- Many rules can be illustrated for polynomials:
  $y(x) = x^3 + a * x^2 + b * x + c => y'(x) = \frac{dy}{dx} = 3x^2 + 2a * x + b$
  Rules: derivative distributes over addition and constant multiples;
  it brings a power down by one, but have to multiply by the prior power:
  Power rule: $\frac{d}{dx} x^n = n * x^{n-1}$, true even for negative n.
- **Product rule**: $\frac{d}{dx}(a(x) * b(x)) = a'(x) * b(x) + a(x)b'(x)$
- **Chain rule**: let $w(y)$ be a function of y (such as the y(x) used above.)
  Then as a function of x:
  $w(y(x)) = sin(y(x)) => \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = cos(y) * (3x^2 + 2a * x + b)$

  To express the latter as a function of x, you still have to expand the y in cos(y).
  You have to know a few special derivatives like $sin' = cos$.
- Quotient rule: $\frac{d}{dx}\frac{a(x)}{b(x)} = \frac{a'b - ab'}{b^2}$
  Prove this from the product, power, and chain rules.
  Hint: $\frac{1}{b(x)} = b(x)^{-1}$
- What if we have a function of multiple variables? Then use **partial derivatives**, where you act as though other variables are constant:
  $z(x, y) = x^3 * y + 3 * x * y^4 => \frac{\partial z}{\partial x} = 3x^2 * y + 3 * y^4$

- Less important for today: the "fundamental theorem" of calculus:
  $\int_a^b f'(x)dx = f(b) - f(a)$
  (This can also be stated as $\int_a^b \frac{df}{dx} dx$, which tempts you to cancel out dx in the fraction. You can't quite do that because this notation means infinitesimals, but the intuition is useful.)

Quick intro/recap: complex numbers

The complex numbers $\mathbb{C}$ are the (two dimensional) field extension of $\mathbb{R}$ formed by adding $i = \sqrt{(-1)}$



Argument φ and modulus r locate a point in the complex plane.

Two ways to think of a complex number z: **rectangular** and **polar**:
$z = Re(z) + i * Im(z) = x + iy = r * e^{i\phi} = r * (cos(\phi) + i * sin(\phi))$
Here r is the length $|z|$ of a complex number, and ϕ is the angle it makes with the positive real axis, and x and y are the real and imaginary parts of z.

- Addition is easy: 2-d vector addition
- Multiplication is new: multiply lengths, but add angles
  - Needham (see reference) calls multiplication an "ampli-twist", because it scales by a factor ("ampli" for amplification), and it rotates ("twist".)

Why a complex derivative is so much more powerful than a real one

A function $f(z) = u(z) + iv(z)$ has real and imaginary parts, and each of these is a function of the real and imaginary parts of $z = x + iy$:
$f(x + iy) = u(x + iy) + iv(x + iy) = u(x, y) + iv(x, y)$

Notice the two ways we can think of u and v:

- as functions of the single complax variable $z = x + iy$
- as functions of two real variables x and y.

In a complex derivative, h can approach 0 from any direction, and this still has to give you the same answer:
$f'(z) = \frac{df}{dz} = lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$

$= lim_{\triangle x\to 0}\frac{f(z+\triangle x)-f(z)}{\triangle x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= lim_{i\triangle y\to 0}\frac{f(z+i\triangle y)-f(z)}{i\triangle y} = \frac{1}{i}(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}) = (-i)\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

(Note the **partial derivative**s in the last two equations.)

Let's work out some of the consequences:

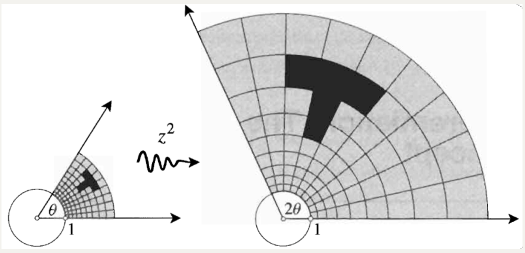
- Cauchy Riemann equations:
  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- Re(f) and Im(f) are each **harmonic**, e. g. $u_{xx} + u_{yy} = 0$
  --> **prove this!**
  (Notation: u_x means $\frac{\partial u}{\partial x}$. So two subscripts means you take two partial derivatives.)
  [Harmonic is a special 2-d case of *Laplace's equation*: $\nabla^2 F = 0$, which is important in many areas of physics]

Question: The "conjugation" function is $\bar{z} = x - iy$
-- Is this differentiable?
-- Is this analytic?

$f(z) = z^2$ in greater depth

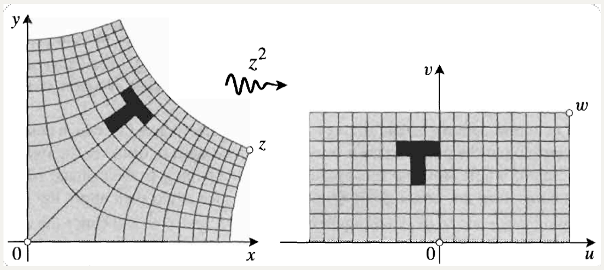
Problem: work out z^2 for $z = a + ib$

Problem: work out z^2 for $z = re^{i\theta}$



Question: Is this conformal as a complex function?
Answer: yes, for every point except for z=0 (and $z = \infty$, if you treat this like a point - which you'll see soon!) Around these points, the map *doubles* angles, and so is not conformal. These are points where the derivative is 0 or undefined.

Problem: what are the sets in $\mathbb{C}$ does z^2 map to to constant u? to constant v?



Question: What are the images of horizontal and vertical lines?

Stop discrimination: ∞ is a number too (well, sort of)

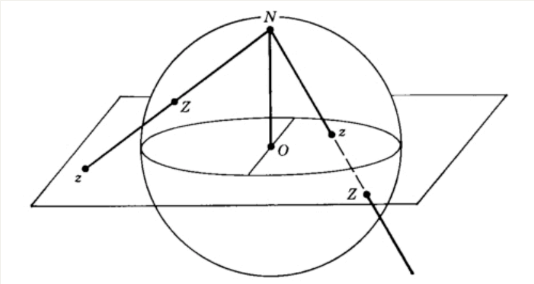
We say $lim_{x->0+}\frac{1}{x} = \infty$, but $lim_{x->0-}\frac{1}{x} = -\infty$
How could this get easier in $\mathbb{C}$, when there are now infinitely many directions from which to approach 0?

Answer: we will see this better geometrically with stereographic projection, where the two limits above are thought to approach the same point ∞ from different directions.

Stereographic projection: useful to understand many complex maps

Stereographic projection is a map SGP from the complex plane $\mathbb{C}$ to the unit 2-sphere $\mathbb{S}^2$ in three dimensions, where we can think of $\mathbb{C}$ going through the equator of $\mathbb{S}^2$, which is just a unit circle ("C"), forming the boundary of the unit disk ("D".)

Map any point z from $\mathbb{C}$ into that point of Z=SGP(z) of $\mathbb{S}^2$ that is on the line from z to N, the "north pole" (0,0,1).



- The area outside the unit circle gets mapped to the upper hemisphere
- The area inside the unit circle gets mapped to the lower hemisphere
- $0 \to$ south pole ("S"), $\infty \to$ north pole N
- The map SGP is 1-1 from $\mathbb{C} \cup \{\infty\}$ onto $\mathbb{S}^2$

With just a bit more reasoning, we can find nice geometric properties of SGP.

Question What sets get mapped onto:

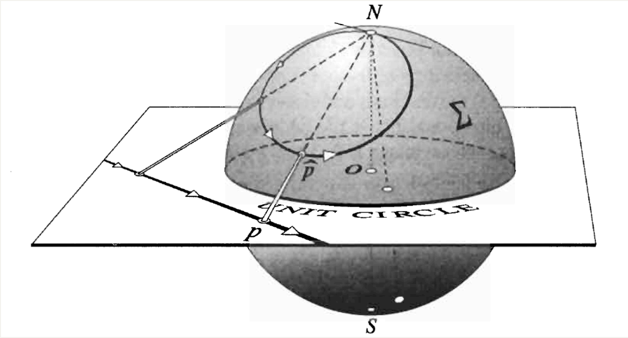
1. Latitude circles
2. Longitude circles
3. Other circles?

Answers:

1. By symmetry, the pre-images of latitude circles are circles centered at the origin in $\mathbb{C}$. Notice that these circles fill $\mathbb{C}$ except for 0. This is another way to see the 1-1 correspondence between $\mathbb{C}$ without 0 and $\mathbb{S}^2$ without both poles.
2. **Prove**: longitude circles are the images of lines through the origin.
3. **Trick question**: "other circles" may be images either of lines (not through the origin) or of circles in $\mathbb{C}$ not centered at 0.

Result: Circles on the sphere correspond to circles and lines in $\mathbb{C}$, depending on whether the circle goes through the north pole. Let's establish this more rigorously.

Lemma: SGP maps lines to circles
Prove, using the following diagram:



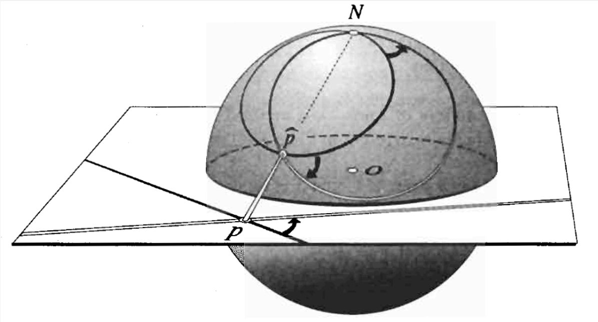
Use this same diagram to prove:

Lemma: the tangent at N to the image circle is parallel to the line

These facts allow us to establish a perhaps surprising result:

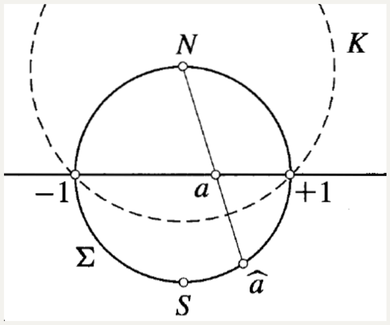
Theorem: SGP is conformal

Prove this using yet another cool diagram from Needham:



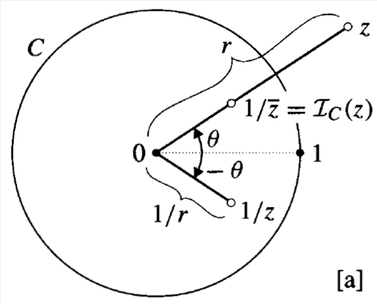
[**Theorem** (not proved fully here, but easy using facts about inversions): SGP can be thought of as the inversion (in $\mathbb{R}^3$) by the circle K centered at N with radius $\sqrt{2}$. That means it also sends circles to circles.]

This is illustrated here, restricted to a plane through +1 (of $\mathbb{C}$), N, and S



So we see that SGP shows much of the geometry of complex maps:

Example 1: The reciprocal function z -> 1/z is a) geometric inversion followed by b) conjugation:



How do these operations look in stereographic image?
Conjugation is easy: it just reflects the sphere in the vertical plane going through the poles and the point 1 (the plane parallel to the image page.)

Inversion is trickier (not fully covered here), but the surprising result is another symmetry between reciprocals:
Corollary: The stereographic image of inversion is reflection in the equatorial plane ($\mathbb{C}$)

Corollary: The stereographic image of the reciprocal map is reflection on horizontal plane followed by reflection on this vertical plane, which amounts to rotation by the angle π around the real axis

Example 2: We have seen how the squaring function $z \to z^2$ looks on the complex plane. But how does it look on the sphere?
Answer: 1) All angles are doubled, as seen "from above", relative to the positive x direction. 2) Points not on the equator are moved toward the nearest pole (N or S)

One very useful family of conformal maps: linear fractional "Möbius" transformations (as much as we have time for today)

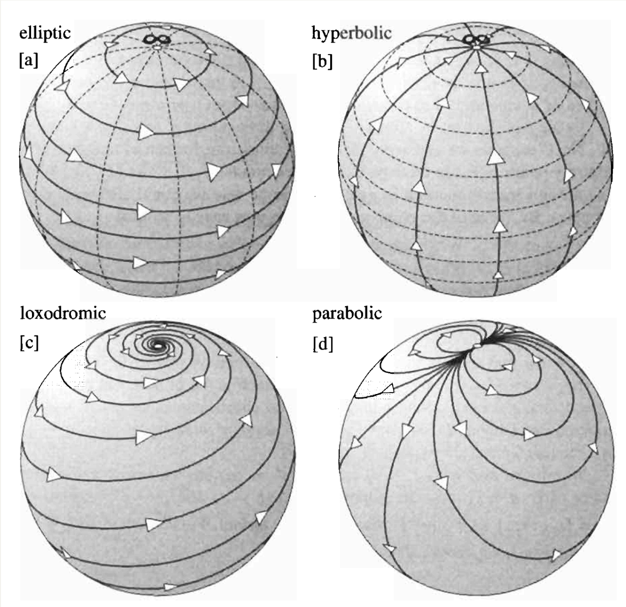
$M(z) = \frac{az+b}{cz+d}$

Examples:

- $z \to az$ (for constant a, Then b=0, c=0, and d=1)
- $z \to z + b$ (a=0, c=0, and d=1)
- $z \to 1/z$ (which we have considered: a=0, b=1, c=1, and d=0)
- $z \to constant$ (a "boring" case, whenever $ad - bc = 0$)
  In this case, what is the constant?

<Draw on whiteboard how these act on $\mathbb{C}$>

Which of these maps are shown by the following "types" of Möbius transformations?



An important result:
Theorem: Every non-constant M(z) is 1-1 and onto, considered as a map between spheres. It is analytic in each direction.
Proof: work out inverse, verifying this works exactly when $ad - bc \neq 0$

Another proof: verify the steps shown below are invertable in this case.

Every Möbius transformation decomposes into the following simpler steps:

(i) $z \mapsto z + \frac{d}{c}$, which is a translation;

(ii) $z \mapsto (1/z)$;

(iii) $z \mapsto -\frac{(ad-bc)}{c^2}z$, which is an expansion and a rotation;

(iv) $z \mapsto z + \frac{a}{c}$, which is another translation.

Theorem: Given three distinct points q, r, s in $\mathbb{C}$ and three distinct points $\hat{q}, \hat{r}, \hat{s}$ in $\mathbb{C}$, there is a unique Möbius transformation that sends $q \to \hat{q}, r \to \hat{r}, s \to \hat{s}$

We'll prove this using:

Lemma: Given three distinct points p, q, r in $\mathbb{C}$, there is a unique Möbius transformation that sends $q \to 0, r \to 1, s \to \infty$

Prove this! (Hint: try building a formula, starting with numerator and denominator, then adjust.)

The **cross ratio** is defined as [z, q, r, s] = $\frac{(z-q)(r-s)}{(z-s)(r-q)}$

You can see that this has the desired effect on q, r, s. You can arrive at this by first noticing the (z-q) and (z-s) factors are needed respectively to have the right image of q and s. Then the two factors with r create a constant multiple so that the image of r is 1.

We look again a the analogy between these maps and 2 by 2 matrices (with complex coefficients):

$$[M] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \widetilde{M}(z) = \frac{az+b}{cz+d} \to M(z)$$

The left map above is in fact a group homomorphism, respecting multiplications:

- Matrix multiplication (on the left)
- Composition of maps $\widetilde{M(z)}$ (in the middle)

Nonsingular matrices (i.e. with determinant $ad - bc \neq 0$) form a group, as do non-constant M(z) (satisfying the same condition.) But note that there are different ways to specify the "same" M(z) (e.g. $\frac{az+b}{cz+d} = \frac{2(az+b)}{2(cz+d)}$) This is what is meant by $\widetilde{M(z)}$: one way to specify a given complex map M(z). To get to the latter, you have to "divide out" by the set of nonzero ad-bc. One convention is to choose the M(z) for which ad-bc = 1.
In this case, we get an **isomorphism** between matrices with determinant 1 and M(z).

There is a lot more we can do with these maps: fixed points, preserved regions, and much more of their geometry! We'll continue with that next week.

The Riemann Mapping Theorem:

Theorem: For any simply connected (non-empty) open subset U of the complex plane, there is an 1-1 analytic map between U and the unit disk D (inside of the unit circle.)

This means we can solve problems (like many differential equations) for such a region U by translating it into D and just solving there!

One important ingredient for this to work is that complex maps preserve functions being harmonic. As a result, turning physics & engineering problems into 2-d makes this a very useful tool

More examples on this next week. For now, you could consider "how many" Möbius maps send one side of a given circle/line to a similar region.

Example: "how many" Möbius maps send the unit disk to itself?

Conclusion

Conspiracy theory: Nature is really complex, but tries to keep us ignorant by making it seem like we live in a universe with real dimensions.
And yet, the math becomes so much cleaner when we use complex functions!

References

Homework for next time: watch this - hopefully understandable after today:
https://www.youtube.com/watch?v=48aerHs9wL0

A good, brief survey of complex analysis appears in the Princeton Companion to Applied Mathematics (2016)
(Their Companion to [pure] Mathematics (2008) doesn't even list this as an article, but contains several references.)

Ancient and more recent books:

-- Knopp, Elements of the Theory of Functions (multiple volumes originally published starting 1926 - cheap old Dover translations available at garage sales)

-- Needham, Visual Complex Analysis, 1997