BMC ADVANCED: GEOMETRY ON THE WALLS

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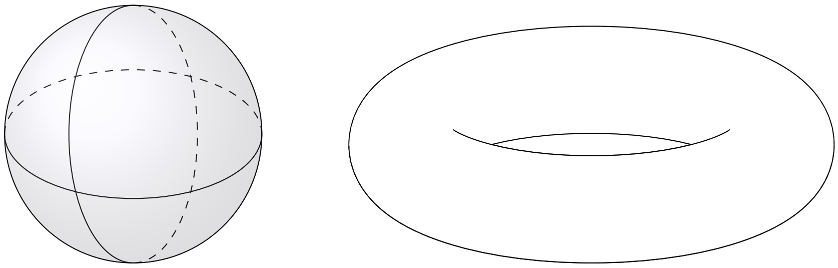
1. Shape and geometry

(1) How can we tell the difference between different shapes? Imagine you were an ant living on the surface of a sphere or a torus. How would these surfaces seem different? How would they seem the same?



(2) Do you know any invariants that could distinguish between these two worlds? How could you calculate them?

2. Wallpaper and bricks

(3) You are an ant living on the wallpaper in my child's room (fig. 1). If the wallpaper extended infinitely in all directions, what would you say about the shape of the world? What if you lived on the shapes in fig. 2?

(4) What if you were an ant living on brickwork? Consider each of the two patterns in fig. 3. What symmetries do you see? What is the smallest repeating unit that you can find (this is called the *fundamental cell*). How do these combine to tile the plane? If the bricks extended infinitely in all directions, what would you say about the shape of the world?

(5) Brickwork can get even fancier. Consider the patterns in fig. 4. What symmetries do you see? What is the fundamental cell? Can you say anything about the shape of these worlds?

3. Orbifolds

(6) Consider the bedspread in fig. 5. Identify the fundamental cell, and examine how they combine to tile the plane. Think carefully about the shape of the world as viewed by an ant living on the bedspread. How can you glue the fundamental cell to itself to give a complete description of pattern? What shape is this?

The orbifold shop will add parts to your orbifold. Their price list is as follows:

- Add a handle \$2.00
- Add a mirror \$1.00
- Add a cross-cap \$1.00
- Add a cone point of order n .... $\$1.00 \times \frac{n-1}{n}$
- Add a corner reflector of order n (must be placed on a mirror) .... $\$0.50 \times \frac{n-1}{n}$

(7) If you start with a sphere and add a mirror, what shape do you end up with? What if you add two mirrors? What if you add a mirror and a cross-cap?

(8) If you had $2.00 to spend at the orbifold shop, what could you buy? How many different orbifolds could you create?

Aside: You can represent an orbifold using *Conway notation*.

- A cone point of order n is represented by writing the number n
- A mirror is represented by *
- A cross-cap is represented by $\times$
- A corner reflector of order n is represented by writing the number n after the * representing the mirror it is on.

For example, an orbifold with four cone points of order 2 would be written 2222. An orbifold with a cone point of order 2, a mirror, and a corner reflector on that mirror of order 3, would be written 2*3.

(9) Each of these orbifolds may or may not actually exist, and may or may not correspond to a wallpaper pattern. How many of these can you construct yourself, thus proving that they really exist?

(10) What would happen if you spent less than $2.00 at the orbifold shop? What would happen if you spent more?

FIGURE 1. My child's wallpaper

(A) Tan Hall

(B) Gate

FIGURE 2. Around Berkeley

(A) Octagonal bricks

(B) Square bricks

FIGURE 3. Brickwork

(A) Offset bricks

(B) Squiggly bricks

(C) Triangular bricks

FIGURE 4. More brickwork

FIGURE 5. Bedspread and cat

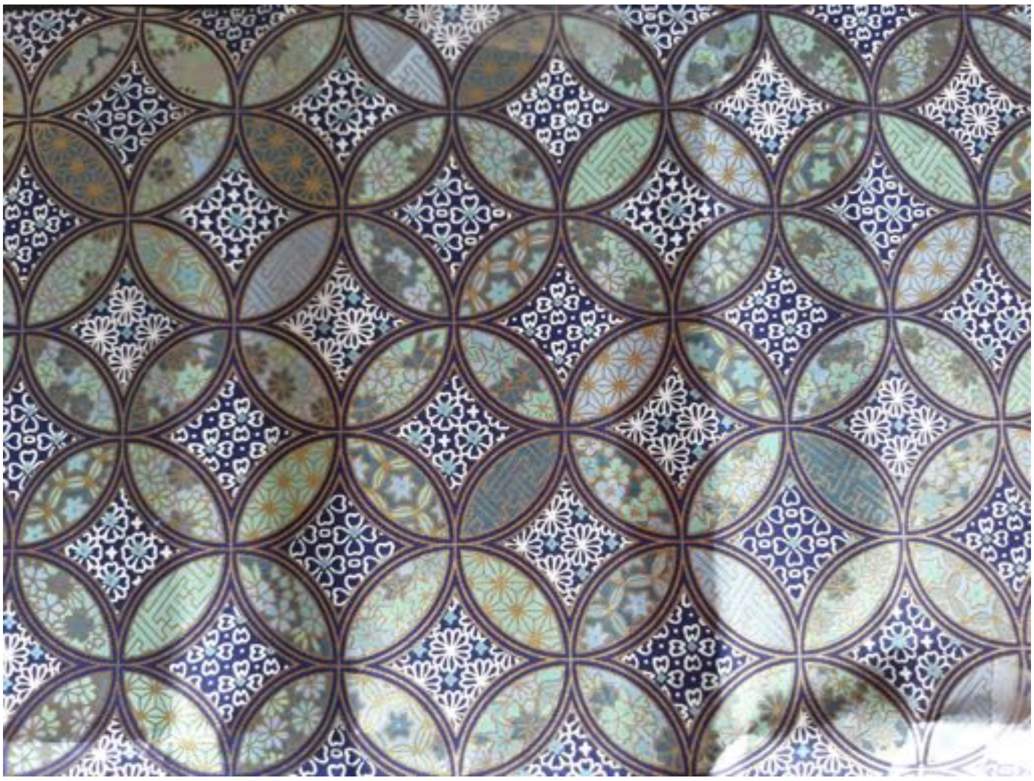


FIGURE 6. Wrapping paper. How many wallpaper patterns can you find within?

Many of these pictures have been taken from the website of Professor Dror Bar-Natan. To see more, visit `http://www.math.utoronto.ca/~drorbn/Gallery/Symmetry/Tilings/`