Welcome back; we'll get started at 6:10pm.

If you solved the HW problem,
find your answer!

Recap of some facts from last time:

- FORMULA FOR A GEOMETRIC SERIES

If $|r| < 1$, then

$$F + Fr + Fr^2 + Fr^3 + \cdots = \frac{F}{1-r} \qquad \left[\frac{\text{first}}{1-\text{ratio}}\right]$$

- KOCH SNOWFLAKE
  - → Perimeter is infinite
  - → Area is finite

- HW: What did you get for the area?

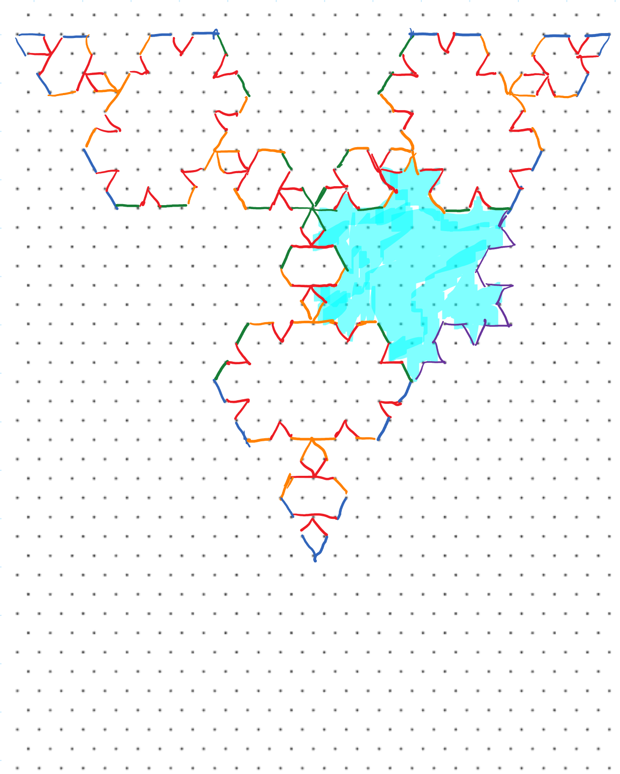
$$\boxed{\text{Total area: } 729 + \underbrace{}_{\substack{\text{w/ } F = 3(9^2) \\ r = 4/9}}^{\text{geom series}}}$$

$$729 + \frac{3(9^2)}{1 - \frac{4}{9}}$$

$$= 729 + 243 \cdot \frac{9}{5}$$

$$= 1166.4$$

Koch anti-snowflake



1st △
$P = 81$
$A = 729$ △

2nd iteration
perimeter $\cdot \frac{4}{3}$
area $- 3(9^2)$
$P = 108$
$A = 486$ △

3rd iteration
perimeter $\cdot 4/3$
area $- 12(3^2)$
$P = 144$
$A = 378$ △

4th iteration
perimeter $\cdot 4/3$
area $- 48(1^2)$
$P = 192$
$A = 330$ △

$192\left(\frac{1}{3}\right)^2$

$$729 - \boxed{3(9^2)} - \boxed{12(3^2)} - \boxed{48(1^2)} - \boxed{} \cdots$$

infinitely many rounds

$$729 - \left[3(9)^2 + 3(9^2)\cdot\frac{4}{9} + 3(9^2)\left(\frac{4}{9}\right)^2 + 3(9^2)\left(\frac{4}{9}\right)^3 + \cdots\right]$$

$\cdot \frac{4}{9}$ $\cdot \frac{4}{9}$ $\cdot \frac{4}{9}$ $\cdot \frac{4}{9}$

$$r = \frac{4}{9}, \quad F = 3(9^2)$$

$$= 729 - \frac{3(9)^2}{1 - \frac{4}{9}}$$

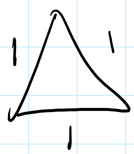
Area of the

$1-\frac{4}{9}$

Area of the

$729 - 243 \cdot \frac{9}{5} = \boxed{291.6}$ anti snowflake

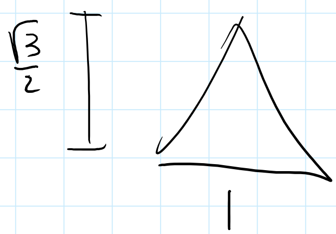
small △'s

area 

$= \frac{\sqrt{3}}{4}$

Actual area

$291.6\left(\frac{\sqrt{3}}{4}\right)$



Why is the harmonic series infinite?

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

$$= \infty$$

add up the first ___ terms	Sum	
1	1	
2	1.5	$+\frac{1}{2}$
3	$1.83\overline{3}$	$+\frac{1}{3}$
4	$2.08\overline{3}$	$+\frac{1}{4}$
5	$2.28\overline{3}$	$+\frac{1}{5}$
6	2.45	$+\frac{1}{6}$
7	2.5928	$+\frac{1}{7}$
8	2.71786	$+\frac{1}{8}$
9	2.82897	$+\frac{1}{9}$
10	2.92897	$+\frac{1}{10}$
11	3.0198	$+\frac{1}{11}$
12	3.10321	
13	3.180133	
14		

14

$$\underbrace{1}_{1} + \underbrace{\tfrac{1}{2}}_{\tfrac{1}{2}} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{> \tfrac{1}{2}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{> \tfrac{1}{2}} +$$

(continued below)

Why?

$$\frac{1}{3} > \frac{1}{4}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$$

Why?

$$\frac{1}{5} > \frac{1}{8}, \quad \frac{1}{6} > \frac{1}{8}, \quad \frac{1}{7} > \frac{1}{8}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

$$+ \underbrace{\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}}_{> \tfrac{8}{16} = \tfrac{1}{2}} + \underbrace{\tfrac{1}{17} + \tfrac{1}{18} + \tfrac{1}{19} + \cdots}_{\text{next } 16 \; > \tfrac{16}{32} = \tfrac{1}{2}} \tfrac{1}{32}$$

$$\underbrace{\tfrac{1}{33} + \cdots \tfrac{1}{64}}_{\text{next } 32 \; > \tfrac{32}{64} = \tfrac{1}{2}} + \cdots$$

Grand total includes infinitely many segments that are larger than 1/2. Sum $= \infty$.

(Diverges.)

Geometric Series Partial Sums

$$F + Fr + Fr^2 + \cdots = \frac{F}{1-r}$$

Proof

$$S = F + Fr + Fr^2 + \cdots$$

$$-rS = -(Fr + Fr^2 + Fr^3 + \cdots)$$

$$S - rS = F$$

$$S(1-r) = F$$

$$S = \frac{F}{1-r}$$

What about a formula for partial sums?

$$T_n = \underbrace{F + \cancel{Fr} + \cancel{Fr^2} + \cancel{Fr^3} + \cdots + \cancel{Fr^{n-1}}}$$

n terms total

$$-rT_n = -\left(\cancel{Fr} + \cancel{Fr^2} + \cancel{Fr^3} + \cdots + \cancel{Fr^{n-1}} + Fr^n\right)$$

$$T_n - rT_n = F - Fr^n$$

$$T_n(1-r) = F(1-r^n)$$

$$\boxed{T_n = \frac{F(1-r^n)}{1-r}}$$

as long as $r \neq 1$.

Partial sum formula

$n = \#$ terms, $F =$ first term, $r =$ ratio, $T_n =$ sum of the first n terms.

Difference of n^{th} powers

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^2 - 1 = (x+1)(x-1)$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$\boxed{1 - x^3 = (1-x)(1+x+x^2)}$$

Partial sum:

$$1 + x + x^2 = \frac{1(1-x^3)}{(1-x)}$$

geom $F=1$

$r=x$

$n=3$

$$(1-x^7) = (1-x)(1+x+x^2+x^3+x^4+x^5+x^6)$$

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6$$

geom $\qquad = \dfrac{1-x^7}{1-x}$

$F = 1$

$r = x$

$n = 7$

More sequence and series problems

Special note: Many of the remaining problems today are taken from textbooks or Alcumus at artofproblemsolving.com.

Note the following relationship between sums of powers of 5 and other powers of 5:

$$\begin{aligned} 5^1 + 5^0 &= 6 \\ 5^2 + 5^1 + 5^0 &= 31 \\ 5^3 + 5^2 + 5^1 + 5^0 &= 156 \end{aligned} \qquad \begin{aligned} 6 \cdot 100 + 25 &= 625 = 5^4 \quad n=4 \\ 31 \cdot 100 + 25 &= 3125 = 5^5 \quad n=5 \\ 156 \cdot 100 + 25 &= 15625 = 5^6 \quad n=6 \end{aligned}$$

Explain the pattern and why it works.

$n=4$: $5^4 = 100(5^0+5^1) + 25$

$n=5$: $5^5 = 100(5^0+5^1+5^2) + 25$

$n=6$: $5^6 = 100(5^0+5^1+5^2+5^3) + 25$

$n=100$ $5^{100} = 100(5^0+5^1+5^2+\cdots+5^{97}) + 25$

general $n \geq 3$ $\boxed{5^n = 100(5^0+5^1+\cdots+5^{n-3}) + 25}$

Let's try to prove this conjecture.

$$5^0 + 5^1 + \cdots + 5^{n-3} = \frac{1(1-5^{n-2})}{1-5} \cdot \frac{(-1)}{(-1)}$$

$$\frac{\;\;\;\;\;\;}{1-5} \quad \cdot(-1)$$

geom $r=5$

$F=1=5^0$

# of terms $= n-2$

$$5^0+5^1+\cdots+5^{n-3} = \frac{5^{n-2}-1}{5-1}$$

$$100(5^0+5^1+\cdots+5^{n-3})+25$$

$$=100\left(\frac{5^{n-2}-1}{4}\right)+25$$

$$=25(5^{n-2}-1)+25$$

$$=25\left[(5^{n-2}\cancel{-1})\cancel{+1}\right]$$

$$=25(5^{n-2})$$

$$= 5^2 \cdot 5^{n-2} = 5^n$$

Multiply by 5 & add 1.

$$(5^0 + 5^1 + 5^2 + \cdots + 5^n)\,5 + 1$$

$$\stackrel{?}{=} 5^0 + 5^1 + 5^2 + \cdots + 5^n + 5^{n+1}$$

Old math contest problems from AoPS Int Alg book

10.73: Source: AHSME

Consider the triangular array of numbers formed when the numbers 0, 1, 2, 3, . . . are placed along the sides and interior numbers are obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown below.

$$\begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 2 & & 2 & & 2 & & & \\ & & 3 & & 4 & & 4 & & 3 & & \\ & 4 & & 7 & & 8 & & 7 & & 4 & \\ 5 & & 11 & & 15 & & 15 & & 11 & & 5 \end{array}$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100?

10.78: Source: AIME

An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has sum 10 times the sum of the original series. Find the common ratio of the original series.

Bonus HW!! If you try these out, remind me to give you answers next time.