Hi! Welcome back. Many of you know already, but I am Kelli Talaska.

Quick poll: What happens when you add up infinitely many numbers?
(Obviously you can't physically do that, but theoretically, if you had infinite time...)

PS: Today is a rare day when you are allowed to use a calculator during my class.

Next, we take a slight detour for a crazy mathematical fact! Sometimes it is possible to add up an infinite list of numbers and get an actual number as a result. Not infinity! Well, sometimes it's infinity. But in very special cases, you get a number.

Let's see some examples, and perhaps you can spot a pattern.

∞ is not a number

- $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots =$ 1

$\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$ $\cdot\frac{1}{2}$

- denominators = power of 2
- always growing as we add more terms, but it's growing faster at the beginning.
- Conjecture: sum is growing arbitrarily close to 1.

Partial sums

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = S_2 + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

$S_5 = \frac{15}{16} + \frac{1}{32} = \frac{31}{32} = \frac{2^5 - 1}{2^5}$

$S_6 = \frac{31}{32} + \frac{1}{64} = \frac{63}{64} = \frac{2^6 - 1}{2^6}$

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

If n is large, this is $1 - \left(\text{tiny } \#\right)$.

$\frac{1}{}$

- $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \cdots =$ $\underline{\frac{1}{2}}$

- terms $\frac{1}{3^n}$
- seems smaller than the last one
  $\Rightarrow$ answer is hopefully between 0 and 1.
- partial sums will grow slower
- conjecture: sum $\frac{1}{2}$, $\frac{2}{3}$, bigger than 1

$S_1 = \frac{1}{3}$

$S_2 = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

$S_3 = \frac{4}{9} + \frac{1}{27} = \frac{13}{27}$

$S_4 = \frac{13}{27} + \frac{1}{81} = \frac{40}{81}$

$S_5 = \frac{40}{81} + \frac{1}{243} = \frac{121}{243}$ $\qquad \frac{1}{2} - \frac{1}{2(3^n)}$

$$S_n = \frac{\frac{1}{2}(3^n - 1)}{3^n} = \cancel{\frac{3^n - 1}{2 \cdot 3^n}} = \frac{3^n}{2(3^n)} - \frac{1}{2 \cdot 3^n}$$

- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} + \frac{1}{4096} + \cdots =$ $\underline{\quad \frac{1}{3} \quad}$

- terms are $\frac{1}{4^n}$

- conjecture for the sum:

$\frac{1}{3}$, $\frac{3}{4}$, $\frac{1}{4}$

$S_1 = \frac{1}{4}$

$S_2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$

$S_3 = \frac{5}{16} + \frac{1}{64} = \frac{21}{64}$

$S_4 = \frac{21}{64} + \frac{1}{256} = \frac{85}{256}$

$S_5 = \frac{85}{} + \frac{1}{} = \frac{341}{}$

$$S_5 = \frac{85}{256} + \frac{1}{1024} = \frac{341}{1024}$$

$$S_n = \frac{\frac{1}{3}(4^n - 1)}{4^n}$$

$$= \frac{4^n - 1}{3(4^n)} = \frac{1}{3} - \frac{1}{3(4^n)}$$

- $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \frac{1}{3125} + \frac{1}{15625} + \cdots = \frac{1}{4}$

- terms $\frac{1}{5^n}$
- guess : verify for HW

$$S_n = \frac{\frac{1}{4}(5^n - 1)}{5^n}$$

- sum $= \frac{1}{4}$

- $\frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \frac{1}{6^5} + \frac{1}{6^6} + \frac{1}{6^7} + \cdots =$ $\frac{1}{5}$

Same ideas

- (Assume $n > 1$.)
  $\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \frac{1}{n^5} + \frac{1}{n^6} + \frac{1}{n^7} + \cdots =$ $\frac{1}{n-1}$

The last one definitely works if you pick an integer n which is bigger than 1. What happens if you pick something stranger, perhaps $\cancel{n = \frac{1}{2}}$ or $n = \sqrt{17}$ – what do you think will happen?

$\sqrt{17}$

$F = \frac{1}{n}$

$r = \frac{1}{n}$

$S = \frac{1/n}{1 - 1/n} \cdot \frac{n}{n}$

$= \frac{1}{n-1}$

⚠ things go wonky:

$n = \frac{1}{2}$

$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \cdots$

$= 2 + 4 + 8 + 16 + 32 + 64 + \cdots$

Sum shooting off to infinity instead of getting close to a specific number.

specific number.

Harder series

When we have an example whose sum is actually a number, we say we have a *convergent series*. Usually it is extremely difficult to predict if your sum will be a number, and if you can figure that out, it can be even harder to find out which number you are supposed to get! Our examples above are called *geometric series* – these are series in which you always multiply by the same amount (called the common ratio) to find the next number in your list. Geometric series have a simple rule for determining whether you get a number or ∞ – check and see if the common ratio is between -1 and 1, not inclusive. If yes, you will get a number when you add up all infinitely many numbers in your list. If no, you will get infinity or something weirder.

Let's see a couple more geometric series– predict whether you will get a number or infinity.

- $\frac{5}{16}+\frac{5}{8}+\frac{5}{4}+\frac{5}{2}+5+10+20+40+80+\cdots=$ ∞
- $\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^5+\left(\frac{2}{3}\right)^6 + \cdots =$ #
- $\frac{1}{\pi}+\left(\frac{1}{\pi}\right)^2+\left(\frac{1}{\pi}\right)^3+\left(\frac{1}{\pi}\right)^4+\left(\frac{1}{\pi}\right)^5+\left(\frac{1}{\pi}\right)^6 + \cdots =$ #

$F = 2/3$, $r = 2/3$ | $S = \frac{2/3}{1-2/3} = 2$

$F = \frac{1}{\pi}$, $r = \frac{1}{\pi}$ | $S = \frac{1/\pi}{1-1/\pi} = \frac{1}{\pi - 1}$

$S_1 = \frac{2}{3} \approx 0.66667$

$S_2 = \frac{2}{3} + \frac{4}{9} = \frac{10}{9} \approx 1.1111$

$S_3 = \frac{10}{9} + \frac{8}{27} = \frac{38}{27} \approx 1.4074$

$S_4 = \frac{38}{27} + \frac{16}{81} = \frac{130}{81} \approx 1.6049$

$S_5 = \frac{130}{81} + \frac{32}{243} = \frac{422}{243} \approx 1.7366$

Series fun facts and geometric series recap

Just for fun, here are a couple more series that are a little mindblowing. These ones are not geometric series. They are called the harmonic series and the alternating harmonic series.

- $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots=\infty$
- $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\ln 2$

fun

The number $\ln 2$ is approximately 0.6931, and $e^{\ln 2}=2$, where e is the famous number you may already be familiar with ($e\approx 2.718$).

Here is a formula for the sum of a **convergent geometric series**. Assume F is the first number (any number), and the common ratio r is between -1 and 1, i.e. $-1<r<1$. Then

$$F+Fr+Fr^2+Fr^3+Fr^4+Fr^5+Fr^6+Fr^7+Fr^8+Fr^9+\cdots=\frac{F}{1-r}$$

You can check that this works on our earlier answers, where we guessed a pattern.

·r ·r ·r ·r ·r

$$\frac{\text{first}}{1-\text{ratio}}$$

give the sum a name: S

$$S = F + Fr + Fr^2 + Fr^3 + Fr^4 + \dots$$

$$-rS = \quad - Fr - Fr^2 - Fr^3 - Fr^4 \quad \dots$$

$$S - rS = F$$

We proved

$$S(1-r) = F$$

$$\boxed{S = \frac{F}{1-r}}$$

that if the sum is a #, this is the # it must be.

We <u>do</u> get a # if

$-1 < r < 1$, and

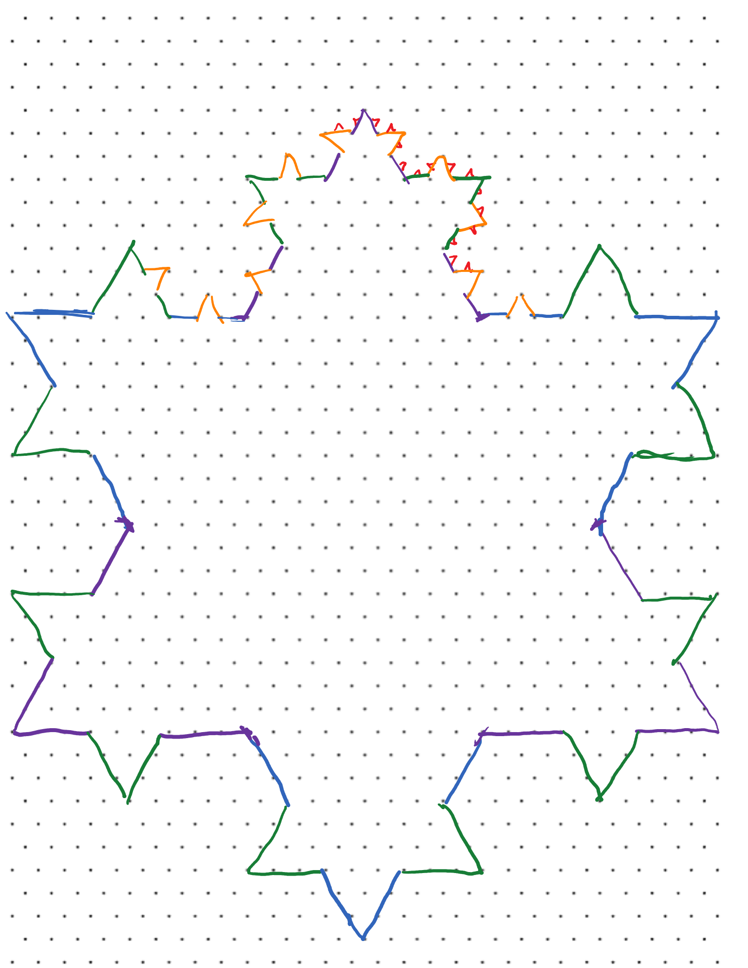
otherwise we don't get a #.

Koch Snowflake

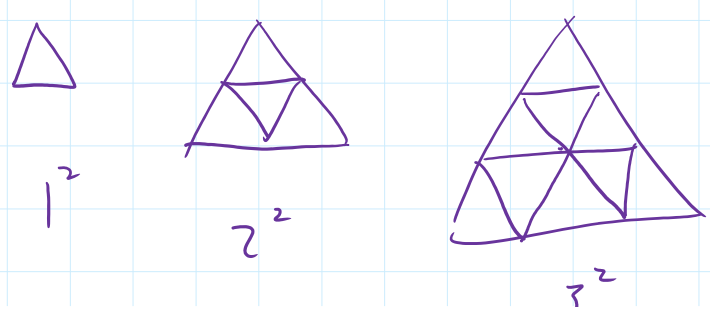
Let's build up the first few iterations of a fractal called the Koch snowflake, and track some data about it as we go. Do your drawings on your triangle dot paper (or whatever you are able to use instead), and keep your data organized in a table on a separate sheet of paper.

We are going to start with a big equilateral with sides of length 27. Technically you can start with any length, but we will be repeatedly splitting into third, so this is convenient for that.

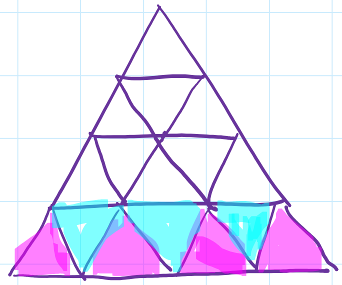
I designed the triangle dot paper so that the wider rows in the short direction (e.g. top and bottom if you orient the paper in portrait mode) have 28 dots, giving us a side length of 27 units.



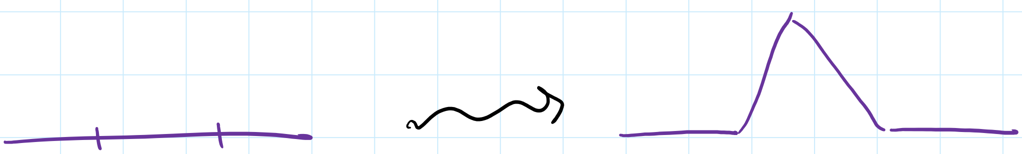
iteration 1
3 sides of 27
iteration 2
12 sides of 9
iteration 3
48 sides of 3
iteration 4
192 sides of 1



3



1
3
5
7



27
$= 3 \cdot 9$

$4 \cdot 9$

If you (theoretically) make the actual fractal with infinitely many iterations, then the

- perimeter will infinite, and
- area will be <u>finite</u> !!!!

Perimeter

iteration	1	2	3	4	...
perimeter	81	$81\left(\frac{4}{3}\right)$ 108	$81\left(\frac{4}{3}\right)^2$ 144	$81\left(\frac{4}{3}\right)^3$ 192	...

+27 +36 +48

Area

Area

iteration	1	2	3	4	5
new area	27^2	$3(9^2)$	$12(3^2)$	$48(1^2)$	$192\left(\frac{1}{3}\right)^2$

weird, ignore

$\cdot 4\left(\frac{1}{9}\right)$ $\cdot 4\left(\frac{1}{9}\right)$ $\cdot 4\left(\frac{1}{9}\right)$

Total area:

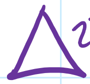
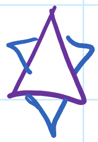
$$729 + \underbrace{}_{\text{geom series}}$$

w/ $F = 3(9^2)$

$r = 4/9$

HW: Total area of the fractal w/ all infinitely

Koch snowflake data

Iteration	Perimeter	Area (# small Δ)
1  27	$27(3) = 81$	$27^2 = 729$
2 	$81 - 3(9) + 6(9) = 108$ $108 = 81\left(\frac{4}{3}\right)$	$729 + 3(9^2) = 972$ $972 = 729 + 729\left(\frac{1}{3}\right)$
3	$81\left(\frac{4}{3}\right)^2 = 144$	$972 + 12(9) = 1080$ $1080 = 729 + 3(81) + 12(9)$
4	$81\left(\frac{4}{3}\right)^3 = 192$	$1080 + 48(1) = 1128$