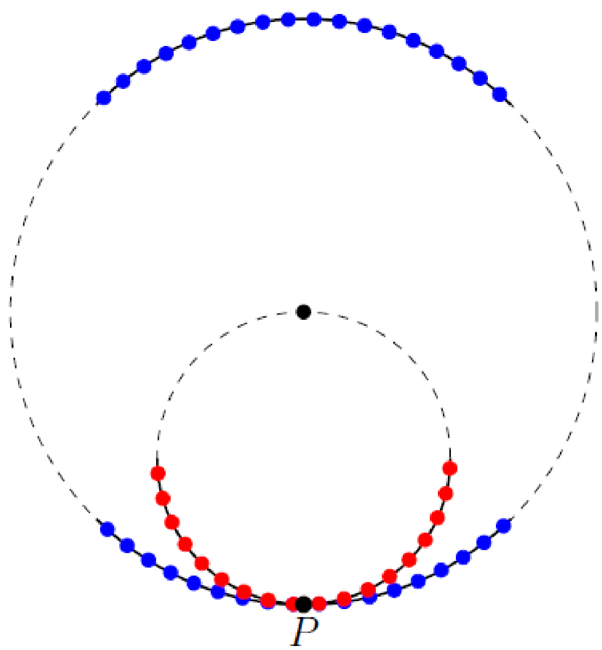
Series II

BMC Int II Spring 2020

April 29, 2020

1 An Approximate Solution

Definition 1.1. For $k \leq N$, let $f_{k,2N}(x)$ denote the amount of light a point P receives from the nearest k points. A picture is shown below.



Exercise 1.2. Prove that

$$f_{2N}(x) - \frac{\pi^2}{N} \leq f_{N,2N}(x) \leq f_{2N}(x)$$

Exercise 1.3. We want to compare $f_{k,2N}(x)$ and $f_{k,4N}(x)$ and we do so in two steps. Prove that the amount of light P receives from all the red points on the inner circle is equal to the amount of light P receives from the blue points on the outer circle.

Exercise 1.4. The amount of light P receives from the red points is $f_{k,2N}(x)$. The amount of light P receives from the blue points on the bottom half is $f_{k,4N}(x)$. Prove that

$$f_{k,2N}(x) - \frac{k\pi^2}{4N^2} \leq f_{k,4N}(x) \leq f_{k,2N(x)}.$$

Exercise 1.5. Prove that for any N and $j \geq 2$ we have

$$f_{k,2N}(x) - \frac{k \cdot \pi^2}{N^2} \cdot \frac{4}{3} \leq f_{k,2^j N}(x) \leq f_{k,2N}(x)$$

Exercise 1.6. Argue that as you take j to infinity for fixed k that

$$\lim_{j\to\infty} f_{k,2^jN} = \sum_{n=-k/2}^{k/2} \frac{1}{(n-x)^2},$$

and conclude that for any N that

$$f_{N,2N}(x) - \frac{\pi^2}{N}\cdot\frac{4}{3} \leq \sum_{n=-N/2}^{N/2} \frac{1}{(n-x)^2} \leq f_{N,2N}(x).$$

Exercise 1.7. Put this together with Exercise 1.2 to show that

$$\lim_{N\to\infty} f_{2^N}(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(n-x)^2}.$$

2 Putting it all Together

Exercise 2.1. Use the approximate solution and exact solution to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)^2} = \frac{\pi^2}{\sin(\pi x)^2}.$$

Exercise 2.2. Plug in $x = \frac{1}{2}$ and prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$