Change (Generating Functions) 0[1]

1. Find a closed form expression for the following three indefinite series.
   a. $1+x+x^2+x^3+x^4+\ldots$
   b. $1+x^2+x^4+x^6+x^8+\ldots$
   c. $1+2x+3x^2+4x^3+5x^4+\ldots$

2. Find the coefficients of x^{14} and x^{17} in the expansion of $(1+x+x^2+x^3+x^4+\ldots)(1+x^2+x^4+x^6+\ldots)$.

3. Interpret the coefficient of x^n in the above product. In other words, complete the sentence, "The coefficient of x^n is the number of ways to _____."

4. Create a product similar to the above expression in which the coefficient of x^n counts the number of ways to write *n* as the sum of a whole number and a positive odd integer.

5. Suppose that a certain country has a 3-cent coin and a 4-cent coin. Then it is not possible to obtain certain amounts of money using only these coins, such as five cents. Determine which amounts can be obtained with these coins.

6. Repeat the previous problem with a 3-cent and 5-cent coin. Then try it once more with a 3-cent and 7-cent coin.

7. Based on the results of the previous problems, make a conjecture as to the largest amount that cannot be obtained using only 3-cent and *b*-cent coins for any value of *b* not divisible by 3. Also conjecture the number of amounts that cannot be obtained.

8. Explain why the situation changes when *b* is divisible by 3.

9. Prove that all amounts greater than or equal to 3*b* can be obtained using 3-cent and *b*-cent coins.

10. Let r_k be the number of ways to obtain exactly *k* cents using 2-cent and 3-cent coins. Make a table listing the values of r_k for $0 \le k \le 15$. What do you notice about this sequence?

11. Prove that $r_{k+6} = r_k + 1$ for all $k \ge 0$.

12. Demonstrate that the series can be written $(1+x^2+x^4+x^6+\ldots)(1+x^3+x^6+x^9+\ldots)$ by suitably interpreting the coefficient of x^k in the product.

13. Let $g(x)$ be the above product. Find a closed form expression for $g(x)$.

14. Show that $g(x)-x^6 g(x)$ and $1+x+x^2+x^3+x^4+\ldots$ are identical except for a single term. Which term is this, and why?

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1 These materials taken from Sam Vandervelde's *Math Circle in a Box*, Chapter 12.