Series I

BMC Int II Spring 2020

April 22, 2020

1 Warm-Up

Exercise 1.1.

1. What is $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots$?

2. What about $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \cdots$?

3. Can you find a pattern for $\sum_{n=1}^{\infty} \frac{1}{r^n}$ for some r?

4. Does this formula work for all r?

Exercise 1.2. Find $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$.

Exercise 1.3. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Exercise 1.4. Using the previous exercise, prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **converges**, i.e. its sum is a finite number and not infinity.

Definition 1.5. The **Riemann zeta-function** is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Remark. You showed what $\zeta(1)$ was and now we will try to find what $\zeta(2)$ is.

Remark. Using a method called analytic continuation, we can define the zeta function for all values of s and show that

$$\zeta(-1) = 1 + 2 + 3 + \cdots = \frac{-1}{12}$$

and

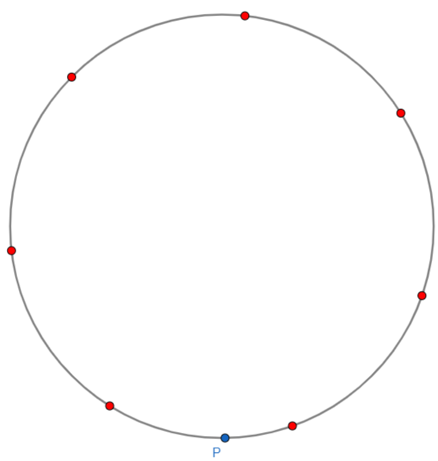
$$\zeta(0) = 1 + 1 + 1 + \cdots = \frac{-1}{2}.$$

Conjecture 1.6 (Riemann Hypothesis)**.** $\zeta(s) = 0$ if and only if $s = -2n$ where $n = 1, 2, 3, \ldots$ or $s = \frac{1}{2} + y\sqrt{-1}$ where y is a real number.

2 An Exact Solution

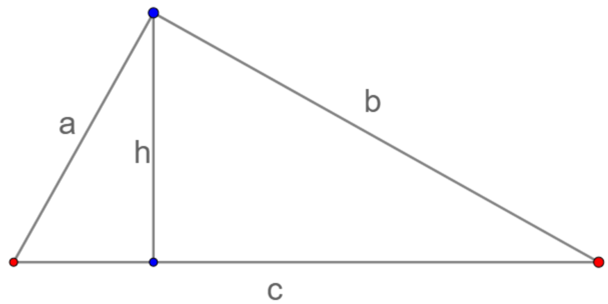
Definition 2.1. A physics fact is that the light you receive at a point falls off as 1 over the distance to the light source squared. So, if we were half as far away from the sun, we would actually get 4 times the light. Let $f_N(x)$ denote how much light you receive if there are N evenly spaced identical light sources on a circle of circumference N, and you are x away from the closest one along the circumference, where $0 < x \leq \frac{1}{2}$.

In the example picture, x is the distance of P from its closest red point and the amount of light received at P is $f_7(x)$. The circle is of circumference 7.



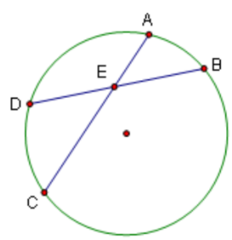
Exercise 2.2. The light received by a source of light d away is $\frac{1}{d^2}$. Prove that $f_1(x) = \dfrac{\pi^2}{\sin(\pi x)^2}$.

Exercise 2.3. In the following picture, prove that $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{h^2}$.

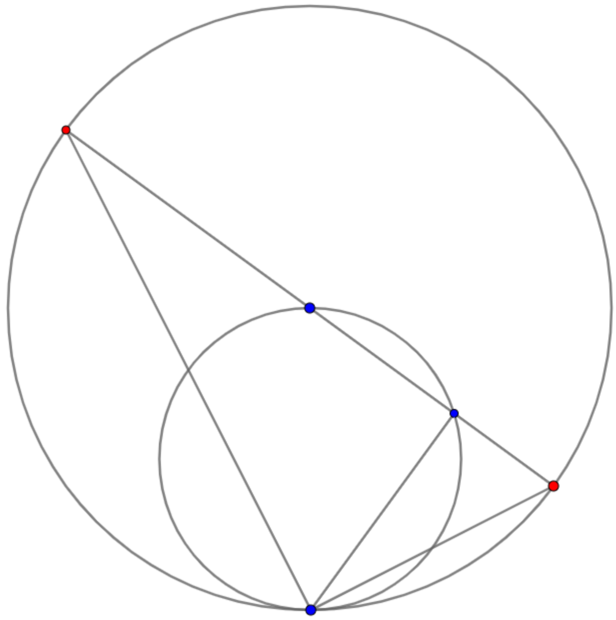


We will need the **power of a point** theorem for the rest of the problems.

Theorem 2.4. If chords AC and BD of a circle intersect inside a point inside the circle E, then $AE \cdot CE = BE \cdot DE$.



Exercise 2.5. Prove that $f_1(x) = f_2(x)$. As a hint, look at the following picture.



Exercise 2.6. Using the same logic as the previous problem, prove that $f_N(x) = f_{2N}(x)$ for all $N \geq 1$.

Exercise 2.7. Put it all together to prove that $f_{2^N}(x) = \dfrac{\pi^2}{\sin(\pi x)^2}$ for all $N \geq 0$.