Mass Point Geometry

BMC Int II Spring 2020

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1 Introduction

Mass point geometry is a method to solve geometry problems involving triangles and asking for ratios of lengths or areas of the triangles. It uses a basic concept from physics but can have some powerful applications.

Archimedes' principle of lever tells us that the center of mass of two points of mass m_1 and m_2 is where

$$m_1 \cdot d_1 = m_2 \cdot d_2,$$

where d_1, d_2 refer to the distance from the center of mass to the points. Note that the center of mass is the balancing point if the two points were placed on a lever. (This is also where Archimedes famously quoted "Give me a place to stand on, and I will move the Earth") A second important property that we will use is that

> The center of mass of a system is not changed by replacing several point masses with their total mass positioned at their center of mass.

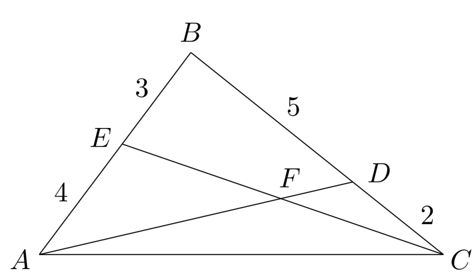
What this means is that we can find the center of mass of several points by successively considering line segments only.

2 Intersecting Cevians

2.1 Example

Definition 2.1. *A* ***cevian*** *is a line segment from a vertex of a triangle to the opposite side.*

The first example of using mass points will be in the first case of two intersecting cevians as in figure 1.



Example 2.2. *In triangle ABC, cevian $\overline{AD}$ and $\overline{CE}$ divide side lengths $\overline{BC}$ and $\overline{AB}$ as shown. Let F be their intersection. What is $\dfrac{CF}{EF} + \dfrac{AF}{DF}$?*

The general strategy will be to pretend to place weights at the vertices A, B, C such that their center of mass is F. Before we begin, we set up some notation.

Definition 2.3. *A* ***mass point*** *is a point P along with a mass m. We write mP to denote that there is a mass of m at point P.*

Definition 2.4. *We can add two mass points mP and nQ. Their sum is the point R between P and Q that is their center of mass. Namely $PR \cdot m = RQ \cdot n$. It has mass $m + n$. So $mP + nQ = (m + n)R$.*

Remark. *The addition we defined is both associative and commutative. So $(xP + yQ) + zR = xP + (yQ + zR)$.*

Now we want to set up the masses so that E is the center of mass of A and B and D is the center of mass of B and C. So we place a mass of $4B$ and hence we get $3A$ and $10C$. Note that we could have just used $1B$ and have fractional masses. Therefore, the weight at E is $3A + 4B = 7E$ and the weight at D is $4B + 10C = 14D$. Finally, the weight at F is $3A + 14D = 17F$. To double check our work and make sure that F is the center of mass, we see that

$$17F = 3A + 4B + 10C = (3A + 4B) + 10C = 3A + (4B + 10C).$$

This means that F is the center of mass of both the line segment $\overline{CE}$ and $\overline{AD}$. So, we use the lever principle one last time to get

$$\frac{CF}{EF} + \frac{AF}{DF} = \frac{7}{10} + \frac{14}{3} = \frac{161}{30}.$$

2.2 Problems

Exercise 2.5. *In triangle ABC, point E is on $\overline{AC}$ so that $CE = 3AE$ and F is on $\overline{AB}$ so that $BF = 3AF$. If $\overline{BE}$ and $\overline{CF}$ intersect at O and the line AO intersects $\overline{BC}$ at D, calculate $\frac{OB}{OE} + \frac{OD}{OA}$.*

Proof. $\frac{OB}{OE} = 4, \frac{OD}{DA} = \frac{3}{2}$. □

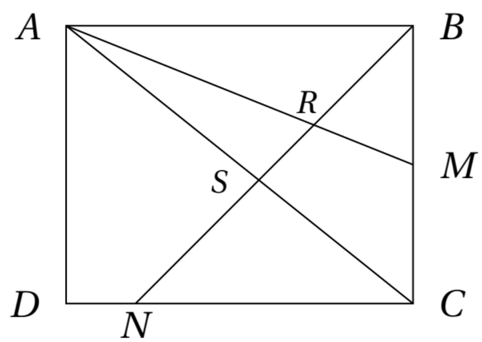
Exercise 2.6. *Let AD be the median bisecting the side BC in triangle ABC. Let O be the midpoint of $\overline{AD}$. Suppose that BO intersects side $\overline{AC}$ at E. What is $\dfrac{AE}{CE}$?*

Proof. $\frac{1}{2}$. □

Exercise 2.7. *Let $\overline{BE}$ and $\overline{CF}$ be medians of triangle ABC and let X be the midpoint of $\overline{BE}$. Suppose that D is on $\overline{BC}$ so that $\overline{AD}$ intersects $\overline{BE}$ at X. Let Z be the intersection of $\overline{CF}$ and $\overline{AD}$. Find $\dfrac{CZ}{ZF}$. (Hint: You will have to use two different mass systems. One assigning masses so that the center of mass is X and then another so that the center of mass is Z.)*

Proof. 4 □

Exercise 2.8. *(Mathcounts) In rectangle $ABCD$, point M is the midpoint of side BC, and point N lies on CD such that $DN : NC = 1 : 4$. Line segment BN intersects AM and AC at points R and S respectively. If $NS : SR : RB = x : y : z$, where x, y, z are positive integers, what is the minimum possible value of $x + y + z$?*



Proof. $x : y : z = 56 : 25 : 45$. □

Exercise 2.9. *(Purple Comet) Triangle ABC has sides with $AB = 39, BC = 57, CA = 70$. Median $\overline{AD}$ is divided into three congruent segments by points E and F and lines BE and BF intersect side AC at points G and H respectively. What is GH?*

Proof. 21. □

Exercise 2.10. *Points D, E, F are on sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively so that*

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2.$$

Let P be the intersection of $\overline{BE}$ and $\overline{AD}$, Q the intersection of $\overline{BE}$ and $\overline{CF}$, and R the intersection of $\overline{AD}, \overline{CF}$. Find the ratio of the areas $\frac{[PQR]}{[ABC]}$.

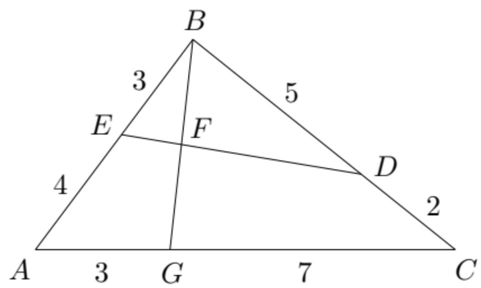
Proof. $\frac{1}{7}$. □

3 Transversals

3.1 Example

Definition 3.1. *A **transversal** is a line that intersects two other distinct lines.*

When dealing with a transversal of a triangle, the point that is on its own side of the transversal will have a split mass.



Example 3.2. *Transversal DE joins points D and E on the sides of triangle ABC so that $AE : EB = 4 : 3$ and $CD : DB = 2 : 5$. Cevian BG divides AC in a ratio of $AG : GC = 3 : 7$. Find $\frac{EF}{FD}$.*

To solve this, we will split up the mass at B so that part of it will balance out the mass at A to have a center of mass at E, and the other part of the mass at B will balance at C at D. We first assign a mass of $21A$ so we have $28B$ and $9C$. Then the contribution at B from C will be $\frac{2}{5}(9C) = \frac{18}{5}B$. Thus, the total mass at G is $21A + 9C = 30G$, the total mass at E is $28B + 21A = 49E$, and at D $9C + 18/5B = 12\frac{3}{5}D$ and at B is $\frac{18}{5}B + 28B = 31\frac{3}{5}B$. To verify that F is indeed the center of mass of the whole system, we see that

$$61\frac{3}{5}F = 21A + 31\frac{3}{5}B + 9C = (21A + 9C) + 31\frac{3}{5}B = 30G + 31\frac{3}{5}B$$
$$= (21A + 28B) + (3\frac{3}{5}B + 9C) = 49E + 12\frac{3}{5}D$$

Thus, we have $\frac{EF}{FD} = \frac{12\frac{3}{5}}{49} = \frac{63}{245} = \frac{9}{35}$.

3.2 Problems

Exercise 3.3. *In triangle ABC, let E be on AB so that $AE : EB = 1 : 3$ and D on BC so that $BD : DC = 2 : 5$ and F on ED so that $EF : FD = 3 : 4$. Let ray BF intersect AC at G. Find $BF : FG$.*

Proof. $\frac{27}{22}$. □

Exercise 3.4. *In triangle ABC, D is the midpoint of BC. Point E lies on AC so that $AE : EC = 11 : 9$ and point F lies on cevian AD so that $AF : FD = 2 : 3$. Ray EF intersects AB at G. What is $GF : FE$?*

Proof. $4 : 7$. □

Exercise 3.5. *(ARML 1992 I8) In triangle ABC, points D and E are on AB and AC respectively. Point T is chosen on BC so that $BT : CT = 2 : 3$ and AT intersects DE at F. If $AD = 1, DB = 3, AE = 2$ and $EC = 4$. What is $AF : FT$?*

Proof. $5 : 13$ □

Exercise 3.6. *(AIME I 2009 4) In parallelogram $ABCD$, point M is on AB so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on AD so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.*

Proof. $\frac{3009}{17} = 177$. □

Problems were taken from the UCLA Math Circle and Tom Rike's excellent handout for the San Jose Math Circle. Additional problems can be found on the AoPS Wiki here: `https://artofproblemsolving.com/wiki/index.php/Mass_points`