Change (Generating Functions) 1[1]

1. Find integers m and n such that $4m+7n=1$. Then find another pair of values for m and n such that $4m+7n=1$ again.

2. If k is a given positive integer, show that we can always find $4m+7n=k$. Then demonstrate that this is still possible if we further require that $0\leq m\leq 6$.

3. Show that the following recipe works for determining whether or not a given amount k can be changed using 4-cent and 7-cent coins. Given k, find integers m and n such that $4m+7n=k$ and $0\leq m\leq 6$. Then k can be changed precisely when $n\geq 0$.

4. Use the idea outlined in the previous problem to determine the largest amount that cannot be obtained using only 4-cent and 7-cent coins.

5. Let a and b be relatively prime positive integers. Generalize the reasoning developed in the preceding problems to analyze the case of two coins worth a cents and b cents. You may use the fact that the Euclidean algorithm guarantees the existence of integers m and n such that $am+bn=1$.

6. Suppose k is an integer between 0 and ab that is not a multiple of a or b. Prove that if the amount k can be changed than $ab - k$ cannot be changed, and conversely if k cannot be changed then $ab - k$ can be changed.

7. Prove that there are exactly $\frac{1}{2}(a-1)(b-1)$ amounts that cannot be changed.

8. Prove that if the positive integers a, b, and c have no common factor then there is some largest amount that cannot be changed using coins worth a, b, and c cents. In other words, show that after some point all amounts can be changed. (We are assuming that a, b, and c are not all divisible by some integer $d\geq 2$ However, any two of them might have a common factor, as is the case for a=6, b=10, and c=15).

1 These materials taken from Sam Vandervelde's *Math Circle in a Box*, Chapter 12.