Expanding Fractions I

BMC Beginner Spring 2020

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1 Introduction

1.1 Some Warm-up Motivation

Exercise 1.1. Now use a calculator to write the decimal expansions of the following fractions:

$$\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$$

and now for

$$\frac{1}{13}, \frac{2}{13}, \ldots, \frac{12}{13}.$$

What patterns do you notice and how do they differ for fractions over 7 and those over 13?

1.2 Modular Arithmetic

We are familiar with modular arithmetic when taking things $\mod 10$. This entails adding or multiplying two numbers, and then taking their remainder when we divide by 10. For example, we have $3 \cdot 8 \equiv 4 \pmod{10}$ because $3 \cdot 8 = 24$ which leaves a remainder of 4 when we divide by 10.

Exercise 1.2. Fill out the following times table using mod 10.

×	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

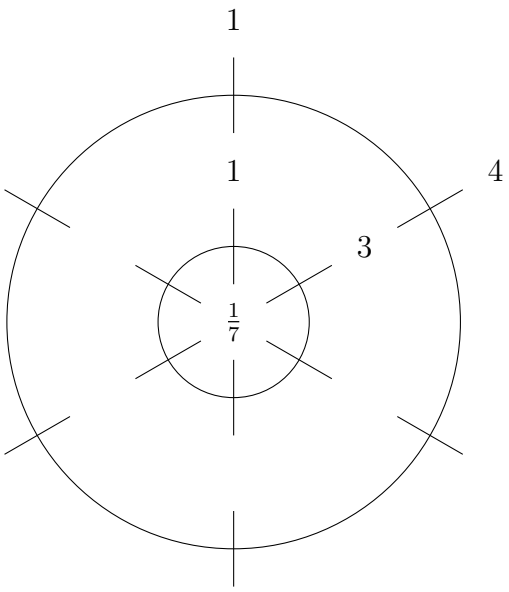
Exercise 1.3. Now fill out the following times table using mod 7 where we taking the remainder after dividing by 7.

×	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

2 Repeating Decimals

Exercise 2.1. Use long division by hand to calculate the decimal representation for $\frac{1}{7}$. From your calculations, can you explain why $\frac{1}{7}$ repeats?

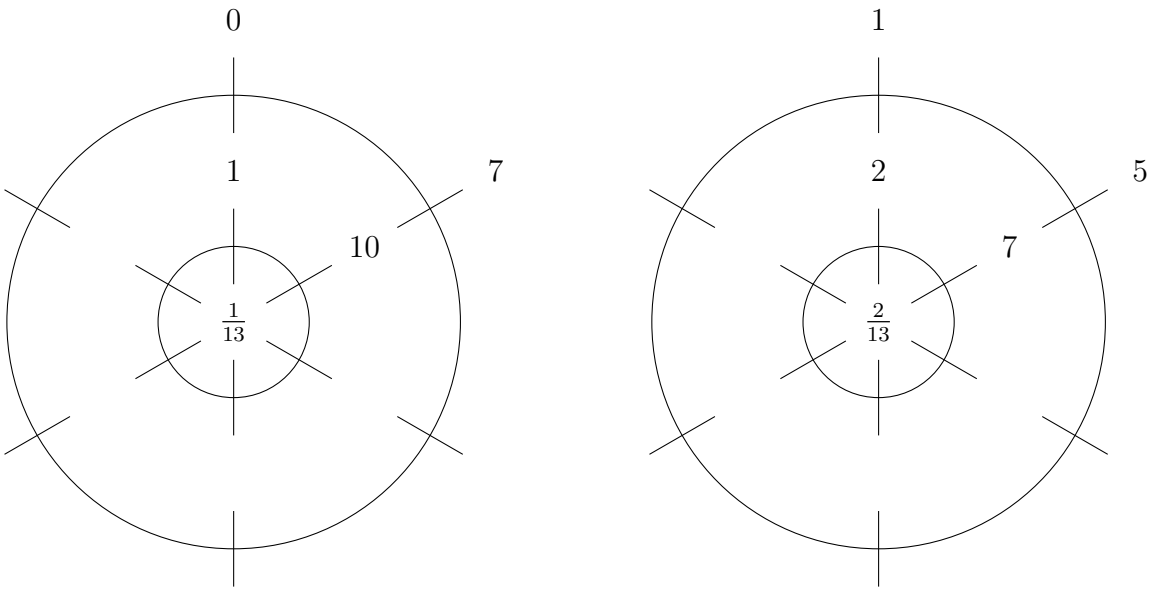
Exercise 2.2. We will create a remainder wheel. The remainders encountered will be placed in the inner circle and the quotient (digit you added to the repeating decimal) will be placed in the outer circle. The first two entries are filled in for you.



Exercise 2.3. What can you learn from the remainder wheel? How can you use it to find the decimal expansion for $\frac{2}{7}$ or $\frac{4}{7}$?

Exercise 2.4. What number do we have to multiply by mod 7 to move clockwise in the inner circle of the remainder wheel?

Exercise 2.5. Now create remainder wheels for dividing by 13. You will need two remainder wheels.



Exercise 2.6. Compare and contrast the observations you made about the $\frac{1}{7}$ wheel with these two wheels. Is the observation in Exercise 2.4 still true? What numbers appear in the inner circle of the two wheels? Will 0 ever appear or will a number ever be repeated? Why do we need two wheels for $\frac{1}{13}$?

Exercise 2.7. Notice that $\frac{1}{3}$ has one repeating digit and 3 divides 9. Moreover $\frac{1}{11} = 0.\overline{09}$ has two repeating digits and 11 divides 99 but not 9. What pattern do you notice? Does the pattern also hold true for $\frac{1}{7}$ and $\frac{1}{13}$? How can we use this pattern to find the number of repeating digits of something like $\frac{1}{37}$ or $\frac{1}{41}$?

These problems were taken from session worksheets held by the Math Teachers Circle. These were held by Elgin Johnston, Steve Pelikan, and Steve Phelps.