Expanding Fractions II

BMC Beginner Spring 2020

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1 Cycles

Exercise 1.1. Use a calculator to write out the decimal expansions of the following fractions.

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{20}.$$

Which of them terminate and which of them have repeating decimals? Can you find a pattern to see which denominators terminate and which do not?

Exercise 1.2. Fill out the following table. Some rows are already filled out to help you.

n	Powers of 10 modulo n	Cycle Length
2	$1, 0, 0$	0
3	$1, 1, \ldots$	1
6	$1, 4, 4, \ldots$	1
7		
8		
9		
11		
13		
17		
19	$1, 10, 5, 12, 6, 3, 11, 15, 17, 18, 9, 14, 7, 13, 16, 8, 4, 2, 1$	18
21		
22		
37		
41		

Exercise 1.3. What do the powers of 10 modulo n have to do with the decimal expansion of $\frac{1}{n}$?

Exercise 1.4. A cyclic number is an integer of length n such that if you multiply it by all the numbers from 1 to $n - 1$, you get the same numbers except rotated by some amount. It is best shown via an example. The smallest cyclic number is 142857 because

$$1 \times 142857 = 142857$$
$$2 \times 142857 = 285714$$
$$3 \times 142857 = 428571$$
$$4 \times 142857 = 571428$$
$$5 \times 142857 = 714285$$
$$6 \times 142857 = 857142$$

Where have you seen this cyclic number before? Find another cyclic number (Hint: look at the first exercise)

Conjecture 1.5 (Artin's Conjecture)**.** There are an infinite number of cyclic numbers.

2 Continued Fractions

Exercise 2.1. One thing to notice is that the fractions we've dealt with always terminate or repeat. Why is this always the case?

Exercise 2.2. Consider the decimal $0.101001000100001\ldots$, with the number of 0s between each 1 increasing. Can this be written as a fraction? Why or why not?

Definition 2.3. A number that cannot be written as a fraction of the form $\frac{p}{q}$ is called **irrational**.

Exercise 2.4. The number $a = \sqrt{2}$ is a number such that $a^2 = 2$. Is a rational or irrational? Use your calculator to find the first terms of the decimal.

Exercise 2.5 (Challenge)**.** Suppose that a is rational and that $a = p/q$. Then $(p/q)^2 = 2$ so $p^2/q^2 = 2$ or $p^2 = 2q^2$. What does that mean about whether p or q is even or odd? Can you prove that a must be irrational?

Decimals are not the only way to represent numbers. Another way is called continued fractions. We will look at simple continued fractions which look like

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}}$$

Exercise 2.6. In order to simplify these continued fractions, start at the bottom and work up. Simplify the following two continued fractions as a single improper fraction:

$$4 + \cfrac{1}{3 + \cfrac{1}{2}} \qquad 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

In order to simplify writing these fractions, we use list notation to write

$$[a_1; a_2, a_3, a_4] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}$$

Exercise 2.7. Express the following as continued fractions. You do not need to simplify the continued fractions.

$$[3; 1, 3, 5] \qquad [1; 1, 1, 1, 2] \qquad [5; 4, 3, 2, 1]$$

Exercise 2.8. Rewrite the following continued fractions in list notations.

$$4 + \cfrac{1}{3 + \cfrac{1}{2}} \qquad 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

Exercise 2.9. Can you always simplify a simple continued fraction into a single fraction? Can it ever be irrational?

Exercise 2.10. Can the infinite continued fraction

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}} = [1; 2, 2, 2, \overline{2}]$$

be represented as a fraction? (Challenge) Can you calculate what it is?

These problems were taken from session worksheets held by the Math Teachers Circle. These were held by Tom Davis, Steve Pelikan, and Steve Phelps.