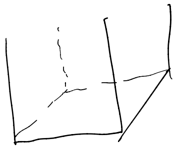
① 2 Mexico 3 Canada 4 US

3 fell asleep.
What is the probability that exactly two of those who fell asleep are from the same country?

Replace the problem with a new one in which we have a box



box is full of pieces of paper

Blindfolded person takes a paper from the box and then opens the eyes and reads the content.

The first piece of paper has the label $\underline{C_1 C_2 C_3}$

The first to fall asleep is C_1

$C_1, C_2, C_3 \mid M_1, M_2 \mid U_1, U_2, U_3, U_4$

$\underline{C_1 C_2 C_3}$, $\underline{C_1 C_3 C_2}$, ...

$C_1 C_2 M_1$, $C_1 M_1 C_2$, ...

The box will have $9\cdot 8\cdot 7$ pieces of paper.
The box is called SAMPLE SPACE.

$S = \left\{ (x,y,z) : \begin{array}{l} x,y,z \text{ are distinct elements of} \\ \{C_1, C_2, C_3, M_1, M_2, U_1, U_2, U_3, U_4\} \end{array} \right\}$

$|S| = 9\cdot 8\cdot 7.$

$\mathbb{P}(C_1 M_3 U_1) = \dfrac{1}{9\cdot 8\cdot 7}$

$D \subseteq S$ D- corresponds to the situation in which exactly two of the sleeping ones are from the same country.

$\mathbb{P}(D) = ?$ Trick: $\mathbb{P}(D^c)$ is easier to calculate

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$D^c = A \cup B$, A - all three from the same country
B - all three from different countries

A, B disjoint
$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

$\mathbb{P}(A) = \dfrac{|A|}{9\cdot 8\cdot 7}$. We need $|A| = ?$

$|A| = |A_C| + |A_M| + |A_U|$, A_C - all three from Canada
A_M - all three from Mexico
A_U - all three from US

$\boxed{\mathbb{P}(A_M) = 0}$

$|A_M| = 0$, $|A_C| = 6$,
$|A_U| = \underline{4}\cdot\underline{3}\cdot\underline{2} = 24$
$|A| = 6 + 24 = 30.$

$|B| = \cancel{24}$
Canadian can be chosen in 3 ways
U.S. person in 4 ways
Mexican person in 2 ways
$4\cdot 3\cdot 2 = 24$
we can order them in 6 ways.

$\boxed{|B| = 24\cdot 6}$

$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) = \dfrac{30}{9\cdot 8\cdot 7} + \dfrac{24\cdot 6}{9\cdot 8\cdot 7} = \dfrac{\overset{15}{\cancel{30}} + 24\cdot\cancel{6}\,3}{9\cdot\overset{4}{\cancel{8}}\cdot 7}$

$= \dfrac{\overset{5}{\cancel{15}} + 3\cdot\cancel{8}}{3\cdot\cancel{4}\cdot 4\cdot 7} = \dfrac{5+24}{12\cdot 7} = \dfrac{29}{84}$

$\Rightarrow \mathbb{P}(D) = 1 - \mathbb{P}(D^c)$
$= 1 - \dfrac{29}{84} = \dfrac{55}{84}$

New solution :

$T = \{ C_1 C_2 C_3,$

$\cancel{\boxed{C_1 C_3 C_2}}$ is illegal thing to write

$C_1 C_3 C_2$

$M_1 \, C_3 \, U_1$ – illegal way to write

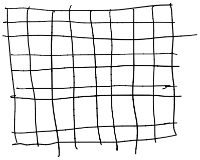
$C_3 M_1 U_1$

$T = \{ (x, y, z);\ x, y, z$ different elements from $\{ C_1, C_2, C_3, M_1, M_2, U_1, U_2, U_3, U_4 \}$ and (x, y, z) is lexicographic order $\}$

$$|T| = \frac{9 \cdot 8 \cdot 7}{6} = 84.$$

. . . .

2. $n \times n$ grid



2 are chosen at random.
Find the smallest n such that the probability that these two squares are adjacent (horizontally or vertically) is $< \frac{1}{2015}$.

Step 1: For fixed n calculate the probability that two randomly chosen squares are adjacent

Step 2: Find the smallest n such that the probability from step 1 is $< \frac{1}{2015}$.

Step 0: what is the sample space S?

$$\begin{matrix} (1,1), & (1,2), & (1,3), & \dots, & (1,n) \\ (2,1), & (2,2), & (2,3), & \dots, & (2,n) \\ (3,1), & (3,2), & (3,3), & \dots, & (3,n) \\ \vdots \\ (n,1), & (n,2), & (n,3), & \dots, & (n,n) \end{matrix} \Bigg\} \text{ labels of tiles.}$$

OUTCOMES $\Leftrightarrow$ elements of the sample space

$\boxed{((1,1),(1,2))} \leftrightarrow ((1,1),(1,3)), ((1,1),(1,4)), \dots$

$\dots ((1,1),(n,n)), \boxed{((1,2),(1,1))}, \dots$

$|S| = n^2 \cdot (n^2-1)$

$$\boxed{S = \{((a,b),(c,d)) : (a,b) \text{ and } (c,d) \text{ are different ordered pairs of integers from } \{1,2,\dots,n\}\}}$$

E – the subset of S that corresponds to two adjacent squares being chosen.

$E = E_H \cup E_V$ $\quad E_H$ – two squares are horizontally adjacent

E_V – vertically adjacent.

Evaluation of $|E_H|$

$|E_H| = |E_H^1| + |E_H^2| + |E_H^3| + \dots + |E_H^n|$

$|E_H^k|$ = # of ways to pick two adjacent squares in the k-th row.

The left square can be chosen in $(n-1)$ ways.
the right one is uniquely determined.

$|E_H^k| = 2(n-1) \Rightarrow |E_H| = n \cdot 2(n-1)$

$\Rightarrow |E_H| = 2n(n-1)$, $|E_V| = 2n(n-1) \Rightarrow$

$$\mathbb{P}(E) = \frac{2 \cdot 2n(n-1)}{n^2(n^2-1)} = \frac{4\cancel{n(n-1)}}{n^{\cancel{2}}\cancel{(n-1)}(n+1)} = \frac{4}{n(n+1)}.$$

The second step: Find minimal n for which

$\frac{4}{n(n+1)} < \frac{1}{2015} \Leftrightarrow \boxed{4 \cdot 2015 < n(n+1)}$

$n(n+1) > 4 \cdot 2015 \Rightarrow \boxed{n = 90}$

$\boxed{n^2+n > 4 \cdot 2015} \Rightarrow n^2+2n+1 > n^2+n > 2015 \cdot 4$

$\Rightarrow (n+1)^2 > 4 \cdot 2015 =$

$\boxed{2^{10} = 1024}$ $\quad 32^2 = 1024$

$64 \cdot 1.4 = 256$

$\quad 64$

89.6

$1024 \cdot 2 = 2048$

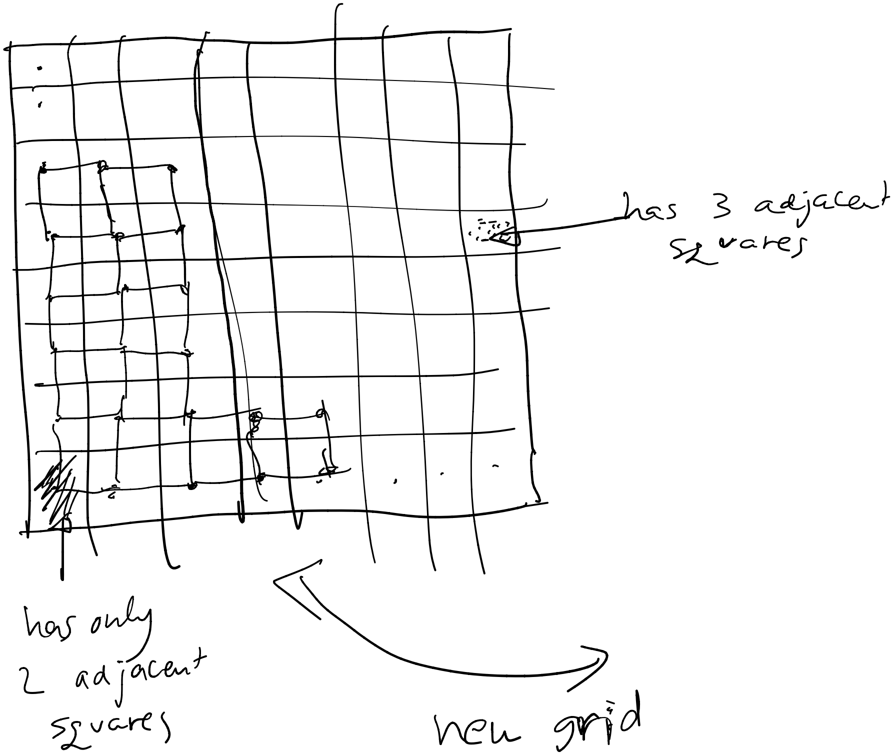
$(n+1)$ somewhere near $2 \cdot \sqrt{2015} \approx 2 \cdot \sqrt{2 \cdot 1024}$

$\approx 2 \cdot \sqrt{2} \cdot 32$

$\approx 1.4 \cdot 64$

If you try $n = 89, 90$, $\boxed{89^2+89 < 4 \cdot 2015}$

No matter what the first square was, there are 4 adjacent squares next to it



(10) 7 dwarfs   Each gets R or G hat.

At the same time each dwarf has to say "R", "G", or keep silent.

They go free if all of those who talked correctly said the color of their hat.

If one or more of them said the color incorrectly, they all loose.

If they are all silent, they also loose.

Is there a good strategy?

With 3 dwarfs there is one strategy that looks good. One dwarf is chosen to be special. That special dwarf is the only one who will guess. The other two remain silent. The probability of survival is 50%

3 Dwarfs

$S = \{RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG\}$

1 2 3

$$\boxed{P(\text{Survival}) = \frac{6}{8} = 75\%}$$

Strategy: If a dwarf sees two friends having the same color hats, the dwarf talks and tells the opposite color.

If the configuration is RRG
1 2 3

The dwarf 3 sees two red hats. The dwarf 3 says "Green". The dwarf 1 remains silent. The dwarf 2 silent.

GRG
1 2 3

1: silent

2: "R"

3: silent

If the configuration is RRR.
1 2 3

Dwarf 1: "Green"
Dwarf 2: "Green"
Dwarf 3: "Green"

Die with a glory

Assume you are one of the dwarfs.
If you see two friends with the same color.
that means the chances of survival are 50%
If you see two friends with different colors, the survival is 100%

(11) The standard deck of 52 cards is properly shuffled. Cards are flipped one by one. You start with \$100. Before each flip you can bet on the next card. If you bet \$$x$ and you guess correctly, you end up with $2x$. If you are wrong, you loose that x dollars. What is the maximal amount of money that you are guaranteed to win?

We can make $\boxed{\$200.}$ Wait until the last card that you know for sure what it will be and bet all the money on that card.

Hint: Start with a deck of 4 cards: 2R and 2G.

4 card game

Wait for the first card. Once it is gone there are 3 cards left. Two have the same color and one is of different color. Without loss of generality, assume that two are R and one is B.

\$100. If we let x dollars on the next card being R.

Case 1: We win; Then we have $100+x$.

Case 2: We loose. Then we have $100-x$ and good news: the remaining two cards are RR.

Case 2: $(100-x)$ on the next card being R we end up with $2(100-x)$, we bet all of $2\cdot(100-x)$ on the next being R. We walk away with $\boxed{4\cdot(100-x)}$.

Case 1: Good news: We won and we have $100+x$. Bad news: the remaining 2 cards are R and B. Skip the next bet. Then bet on the last card and walk away with $\boxed{2\cdot(100+x)}$

$$4\cdot(100-x) = 2(100+x) \iff$$

$$400-4x = 200+2x \iff$$

$$200 = 6x \iff \quad x = \frac{200}{6} = \frac{100}{3}.$$

There is a strategy that guarantees that we will make

$$2\cdot\left(100+\frac{100}{3}\right) = 2\cdot\frac{400}{3} = \frac{800}{3}$$

$$= \boxed{266.66}.$$