Pulling Rabbits from Hats (Conditional Probability)

For the next couple weeks, we'll be working on counting and probability and working up to some pretty fancy stuff, including conditional probability.

First we will warm up with some basic probability.

Use the following info for the entire first block of problems. **Suppose I have a magic hat containing a total of 9 rabbits. You know that 4 of the rabbits are spotted (Samantha, Susan, Stephanie, and Sfluffy), 3 of the rabbits are white (Wabbit, Wobbit, and Wes), and 2 of the rabbits are blue (Bunny and Ben).**

1. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it is a white rabbit?

2. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it has a T in its name?

3. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it has a A in its name?

4. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it has a T or an A in its name?

Use the following info for the next block of problems. **Currently Ben and Bunny are at a special event, so only the remaining 7 rabbits are in the hat. Think about the probabilities we computed above. Should they increase, decrease, or stay the same if Ben and Bunny are not in the hat?**

5. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it is a white rabbit?

6. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it has a T in its name?

7. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it has a A in its name?

8. If I randomly pull one rabbit from the hat (and all are equally likely), what is the probability that it has a T or an A in its name?

The problems we just did were actually something called conditional probability. When I said Ben and Bunny were not in the hat, I told you something about the outcome. Once you knew a little bit about who could or could not be pulled from the hat, perhaps your perspective changed a little, and you knew there was a better or worse chance of something happening. We considered a smaller set of possible outcomes and recomputed our probabilities.

Next, let's consider a fancier problem. Instead of pulling one rabbit from the hat, I will pull two at once. (There is no first rabbit or second rabbit; they both come out at the same time.)

9. How many possible outcomes are there if all 9 rabbits are in the hat? List the options, grouping them as I show you on the board. (If you want to save a little writing, abbreviate the names as "Sa, Su, St, Sf".) We want to be very organized about this, because we will use this list of options to do more problems.

10. I randomly pull two rabbits from the hat with all 9 rabbits in it. What's the probability that at least one of the rabbits is blue?

11. I randomly pull two rabbits from the hat with all 9 rabbits in it. Currently, they are behind a small curtain, so you cannot see which rabbits I got yet. However, I give you a little clue – neither rabbit is white – and I ask, "Now, what's the probability that at least one of the rabbits is blue?" (Before computing it, try to decide if it will be larger or smaller than our answer to #10.)

12. I randomly pull two rabbits from the hat with all 9 rabbits in it. Currently, they are behind a small curtain, so you cannot see which rabbits I got yet. This time I give you a different clue – one of the rabbits is Sfluffy – and I ask, "Now, what's the probability that at least one of the rabbits is blue?" (Before computing it, try to decide if it will be larger or smaller than our answer to #10.)

13. You've done so well on the previous problems that you get to be my magician's assistant now and give the clues. What is a different clue you could give that would make someone MORE confident (but not totally sure) that at least one of the two rabbits pulled from the hat is blue? Compute the new probability given your clue.

14. What is a different clue you could give that would make someone LESS confident (but not totally sure) that at least one of the two rabbits pulled from the hat is blue? Compute the new probability given your clue.

There is a very nice formula for conditional probabilities. Suppose A and B are two *events* (this is official vocab – an event is a subset of the outcomes with a specified property). If we wish to know how likely A is, and we are told that B has definitely happened, the conditional probability is:

$$Prob(\text{A happens, given that B happens}) = \frac{Prob(\text{A and B both happen})}{Prob(\text{B happens})},$$

or, as mathematicians and statisticians would write:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Let's go back and try our new formula on our previous problems to see if we get the same answers. (If we do everything right, we should!)

15. (11.) I give you a little clue – neither rabbit is white – and I ask, "Now, what's the probability that at least one of the rabbits is blue?"

16. (12.) I give you a different clue – one of the rabbits is Sfluffy – and I ask, "Now, what's the probability that at least one of the rabbits is blue?"

17. (13.) Check your probability from #13 using our new formula.

Some extra problems for those who are very quick. You may silently work on these if you are ahead of the group.

18. From 2011 AMC 10: Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

19. Not as hard: We flip a fair coin 10 times. What is the probability that we get heads in at least 6 of the 10 flips?

20. Conditional probability: We flip 10 fair coins, all lined up in a row. You cannot see the outcome yet, but I inform you that the number of heads is even. What is the probability that we get heads in at least 6 of the 10 flips?

21. Conditional probability: A box contains six cards. Three of the cards are black on both sides, one card is black on one side and red on the other, and two of the cards are red on both sides. You pick a card uniformly at random from the box and look at a random side. Given that the side you see is red, what is the probability that the other side is red?

22. Coin A is tossed three times and coin B is tossed two times. What is the probability that more heads are tossed using coin A than using coin B?