Problems and puzzles in probability

Ivan Matić

1. Nine delegates participated in a conference. Two of them were from Mexico, three from Canada, and four from the United States. Three of the delegates fell asleep during the conference. What is the probability that exactly two of those who fell asleep were from the same country?

   Remark: This problem was from AIME 2015.

2. Two unit squares are selected at random without replacement from an $n \times n$ grid of unit squares. Find the smallest positive integer n such that the probability that the two selected squares are horizontally or vertically adjacent is less than $\frac{1}{2015}$.

   Remark: This problem was from AIME 2015.

3. Five red and five green cards are shuffled and top five cards are laid out in a row. What is the probability that all red cards are adjacent and all green cards are adjacent? (The rows *RRRGG* and *GGRRR* satisfy the property that all red and all green cards are adjacent. The row *RRRGR* does not satisfy the property.)

   Remark: This problem was from AIME 2018.

4. There are M green apples and N red apples in a basket. We take apples out randomly one by one until all the apples left in the basket are red. What is the probability that at the moment we stop the basket is empty?

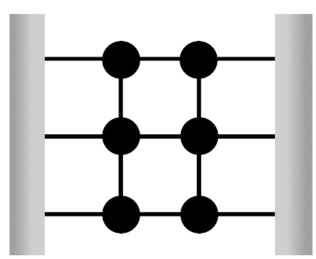
5. There are 1000 green balls and 3000 red balls in container A, and 3000 green balls and 1000 red balls in container B. You take half of the balls from A at random and transfer them to B. Then you take one ball from B at random. What is the probability that this ball is green?

6. A robot performs coin tossing. It is poorly designed, and it produces a lot of sounds, lights, and vapors, and it takes one hour to toss a coin. Yet in the end, when the coin finally lands, it somehow has equal probability of showing heads and tails.

   Two scientists, A and B, enjoy observing this robot and, by analyzing its unusual and faulty behavior, they became fairly decent at guessing whether the coin will land heads or tails half an hour before the coin is released from the robot's hand. The scientist A has 80% chance of successfully predicting the outcome, while the scientist B is successful 60% of the time.

   The robot started its routine, and the scientist A predicts the coin will land tails. The scientist B predicts the coin will land heads. Can you calculate the probability that the coin will land heads?

7. Six islands are connected with two banks of the river with 13 bridges as shown in the picture below.

   When the flood occurs each bridge is independently destroyed with probability $\frac{1}{2}$. What is the probability that it will be possible to cross the river after the flood using the bridges that remain?

8. Two rooms A and B initially contain 1 person. Each second a new person arrives to one of the rooms. If there are a people in A and b people in B the person arriving will choose to go to room A with probability $\frac{a}{a+b}$ or to room B with probability $\frac{b}{a+b}$.

   Determine the distribution of the random variable $\min\{A_n, B_n\}$ where A_n is the number of people in the room A and B_n is the number of people in the room B after n seconds.

9. Evil Commander took the cell-phones from all of his one hundred soldiers. He then correctly wrote the names of soldiers on the phones, but intentionally placed phones randomly in boxes labeled by 1, 2, …, 100. One by one the soldiers are taken to the room with boxes. Once in the room, a soldier is allowed to perform the following 3-step procedure at most 50 times:

    - Step 1. Choose one of the boxes;
    - Step 2. Open the box;
    - Step 3. If the box contains the soldier's own cell-phone, the soldier uses the fingerprint technology to unlock it. Then he can send a message to the President voicing the discontent with Evil Commander.

    After repeating the procedure at most 50 times, the soldier must close all boxes and leave the room without taking any phones regardless whether the soldier succeededs in finding his/her own device.

    However, if the President receives 100 messages (one from each soldier), then the President will force the Evil Commander to return the phones to the soldiers. However, if at least one of the soldiers fails to find the phone in 50 attempts or fewer, then the President will believe to Evil Commander who will deny any mischief and none of the soldiers will get the phone back.

    The night before the game starts, the soldiers are allowed to discuss and make the strategy. Prove that there is a strategy that results in success with probability more than $\frac{1}{4}$.

10. Seven dwarfs are captured by the evil queen who decided to play the following game. The queen puts a red hat or a green hat on the head of each of the dwarfs. The hats are chosen randomly and every configuration is equally likely. The dwarfs can see all the hats except for their own.

    At a signal each dwarf can stay silent, or guess the color of his hat. The queen will free all seven dwarfs if at least one dwarf guesses his hat correctly and no one guesses the hat incorrectly. If all the dwarfs are silent, or some dwarfs say incorrect color, then the queen cooks and eats all of the dwarfs.

    Before the game starts, the dwarfs could decide on a strategy. Prove that there is a strategy that can result in freedom with probability higher than 85%.