How many points are on a plane?

Euclid's axioms

1. **Parallels**: Can't have two different lines through two points and one always exists.

2. A line is defined by two points.

- Can't have 2 different lines through two points
- And one always exists

**Definition** A **plane** is a set, with elements called **points**, with special subsets called **lines** which satisfy the following properties.

A1. **(Incidence Axiom)** Every two distinct points belong to a unique line.

A2. **(Parallel Axiom)** For every line and a point not on it there is a unique line containing this point parallel to the given line.

A3. **(Dimension Axiom)** There exist 3 points which are not collinear.

**Ex. 1.**

Not a plane: 5 points, 1 line

**Ex. 2.**

Problem: lines \( l_1 \) and \( l_2 \) are \( \perp \) to \( l_1 \) and pass through \( A \). So A2 uniqueness doesn't hold.
\[ E_{43} \]

\[ 4 \text{ pts} \quad \text{6 lines} \]

\[ \text{It is indeed a plane!} \]
\[ \text{(smallest that exists)} \]

so 4 pts can be on a plane

**Th.1**

Every line has at least 2 pts

**Pf.**

Assume that \( l \) has only 1 pt \( A \)

\[ \exists \text{ line } l \quad l, C \text{ exists } \quad \text{1} \]

\[ l_1 = AB \]

\[ l_2 \parallel l_1 \quad l_2 \text{ contains } C \]

\[ \Rightarrow \quad \text{Both } l \text{ and } l_1 \text{ are } \parallel l_2 \text{ and pass through} \]

\[ \Rightarrow \quad \text{violation of uniqueness in } A2 \]

Similarly, no empty lines \( \Rightarrow \) all lines must have at least 2 pts.

**Th.2**

All lines have same number of pts.

**Pf.**

Take two lines \( l_1 \) and \( l_2 \), assume \( l_1 \) and \( l_2 \) pass \( A \)

\[ l_1 \parallel PQ \quad \text{1 passes through } X \text{ on } l \]

\[ \text{the } l \text{ intersects } l_2 \text{ (only violates A2)} \]

\[ l_1 \parallel l_2 \]

\[ l_1 \parallel \]
How many points are on a plane?

1, 3, 10, 9, 14, 11, 15 8, 3 4 7, 2 6 8, 1 7, 2 9

Fact: Any two cards belong to a unique set.
How many sets are here?

12 sets. Why no more?

And in fact, this is a 9pt plane with points = cards, lines = sets.

Baby set with only 2 "variables" left, i.e., it is 2-dimensional.

So the big set is a 4-dim space.
Questions:

1. Can there be a plane with 5, 6, 7, 8 points? (No! But why?)
2. Show that if a line has n points, then the plane has n².
3. Can we have 16 or 25 points?
4. How about 36?