Int II Invariants 11/4/20

Wednesday, November 4, 2020 5:08 PM

Invariants II

BMC Int II Fall 2020

November 4, 2020

1 Warm Up Games

1. Place 5 coins heads up on a table. At any moment, we can choose any 2 coins and flip them. Is it possible to get all coins to have tails showing up?
2. We start with four 0's and six 1's written on a board. We can cross out any two of the digits. If the digits crossed out are the same, we write a new 0, otherwise a 1. Which number will be left in the end? Repeat this if there are seven 0's and three 1's.
3. We play another game with four 0's, five 1's, and six 2's written on a board. We can cross out any three of the digits. Then, we replace the three digits with the remainder when we divide their sum by 3. Which number will be left in the end? Repeat this if there are three 0's, four 1's and five 2's.
4. We write the numbers from 1 to 100 on the board. We can cross out two numbers and then write the absolute value of their difference. Is it possible for us to end with only the number 1?
5. We have 13 green, 15 red, and 17 yellow chameleons in the wild. Whenever two chameleons of different color meet, they both change into the third color. Is it possible to have all chameleons of the same color?
6. Three witches are hovering over Berkeley, always keeping at the same height. At every instant, only one of them can move; she can go as far as she wants, but only in a direction parallel to the line connecting her two sisters. If the first witch starts out directly over Evans Hall, the second one 2 miles north, and the third 4 miles east, is it possible that after some time they will end up with the first witch again over Evans, the second one 3 miles northeast, and the third 3 miles southeast?

2 Stomp

In the game Stomp (also known as Lights-Out), you are given a Stomp-piece and allowed to place it anywhere within a grid on each move, as long as it lines up with the grid and stays within the

H H H H H

H T T H H

T T T T H

T T H T T

Parity: the evenness/or oddness of a number.

No, it's not possible to get all coins face up because the parity of the # tails is preserved.

Originally we had 0 = even tails, so # tails is always even

Therefore we can't end up w/ 5 = odd tails.

2) ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ 0

~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~

0 ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~ ~~0~~

~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~

	# 0's	# 1's
0,1	4	6
-0,0 +0	3	6
-1,1 +0	2	6
-0,0 +0	3	4
-1,1 +0	2	4
-1,0 +1	3	2
-0,0 +0	2	2
-0,1 +1	1	2
-1,1 +0	0	2
→	1	0

Claim The parity of the # 1's is preserved.

Options

Remove 0,0 Get 0 Net -0 1s

Remove 0,1 Get 1 Net -0 1's

Remove 1,1 Get 0 Net -2 1's

Parity is preserved

Even initially ⇒ even # 1's at the end ⇒ 0 1's at the end

What will be the last # if start w/ 99 0's, 99 1's.

We start w/ odd # 1's ⇒ end w/ odd # 1's ⇒ end w/ one 1.

3) ~~0~~ ~~0~~ ~~0~~ ~~0~~ 0 ~~0~~

~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~ ~~1~~

~~2~~ ~~2~~ ~~2~~ ~~2~~ ~~2~~ ~~2~~ ~~2~~ 2 ~~2~~ 2.

2,2,1 → 2

0,1,2 → 0

	# 0's	# 1's	# 2's	Sum	
-0,1,2+0	4	5	6	17	-3
-0,1,2+0	4	4	5	14	-3
-0,1,2+0	4	3	4	11	-3
-0,1,2+0	4	2	3	8	-3
-0,1,2+0	4	1	2	5	-3
-0,0,2+2	4	0	1	2	-0
-0,0,2+2	2	0	1	2	-0
	0	0	1	2	

-3 #s + 1 # = -2 #

0 0 0

1 1 1 1

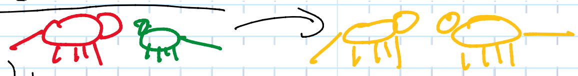
2 2 2 2 2

	# 0's	# 1's	# 2's	Sum	
-2,2,2 +0	3	4	5	14	-6
-2,2,1+2	4	4	2	8	-3
-1,1,1 +0	4	3	1	5	
-0,0,0 +0	5	0	1	2	
-0,0,0 +0	3	0	1	2	

total	r	g	y	y-g

total	r	g	y	y-g
45	13	15	17	2
	0	2	43	41) +39
	4	0	41	41
	0	8	37	29) -12
	16	0	29	29
	0	32	13	-19) -42
	26 / 25	19 / 18	0 / 2	-19 / -16) +3
	7	0	38	38) +54

2 different colors → 2 of the third color



13	15	18	3
15	14	17	3) -3
14	16	16	0
46	0	0	0

26	19	0	-19) +3
25	18	2	-16) +3
24	17	4	-13) +3
23	16	6	-10

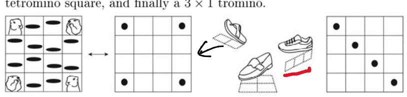
-1,1,1 +0	4	3	1	5) -3
-0,0,0 +0	5	0	1	2
-0,0,0 +0	3	0	1	2
	1	0		2
-0,0,0 +0	3	4	5	14
-0,1,2 +0	1	4	5	14
-0,1,2 +0	1	3	4	1
-2,2,2 +0	1	2	3	8
-0,0,1 +1	2	2	0	2
	0	2	0	2

All the chameleons can end up in the same color iff

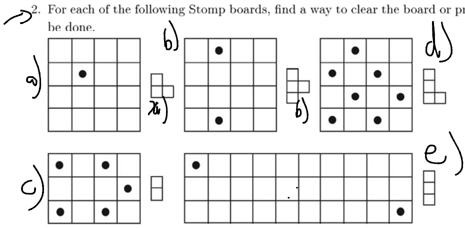
y-g is div by 3 or y-r is div by 3 or g-r is div by 3.

boundaries. In the grid, some squares have dots in them. All squares covered by the Stomp piece change state so that dots covered disappear while empty squares that are covered gain a new dot. The goal is to remove all the dots from the grid, or prove that it is impossible to do so.

1. For each of the following puzzles, try to solve the board using a 1×2 domino piece, a 2×2 tetromino square, and finally a 3×1 tromino.

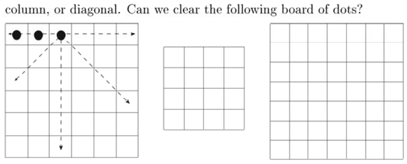


2. For each of the following Stomp boards, find a way to clear the board or prove that it cannot be done.

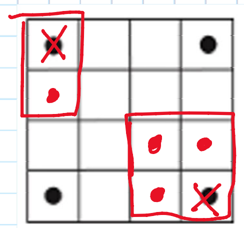


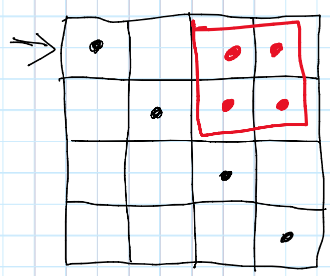
3. Using a T-tetromino on a 3×3 board with a single dot in the top middle square, show how it is possible to end up with a single dot in the central square.

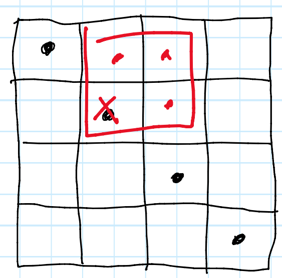
4. Now imagine a variant where we can reverse the state of all the squares in a single row, column, or diagonal. Can we clear the following board of dots?



5. Find all subsets S whose parity of dots is invariant in the Gopher Gun problem and find all attainable configurations of dots.





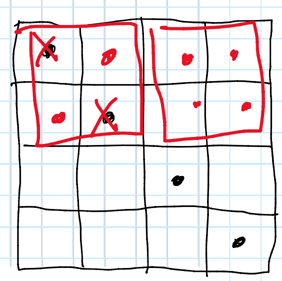


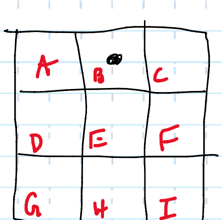
dots in first row will always be odd. ⇒ Parity preserved

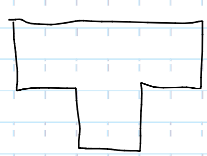
will touch 2 squares or 0 squares

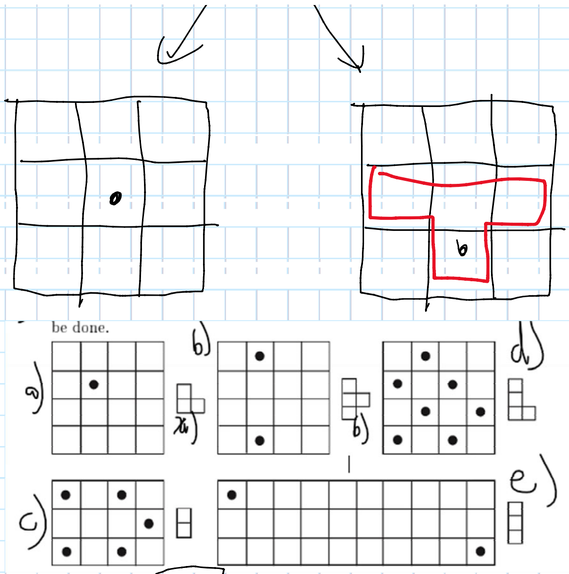
+2, +1-1=0, -2

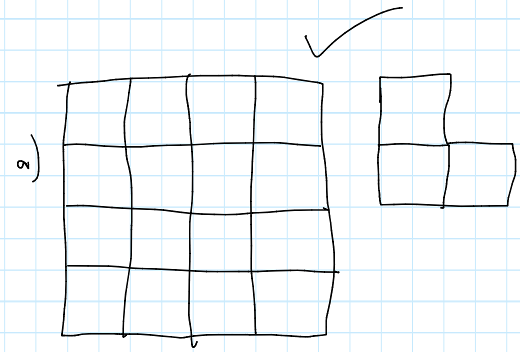
⇒ Impossible to clear this board w/ 2×2



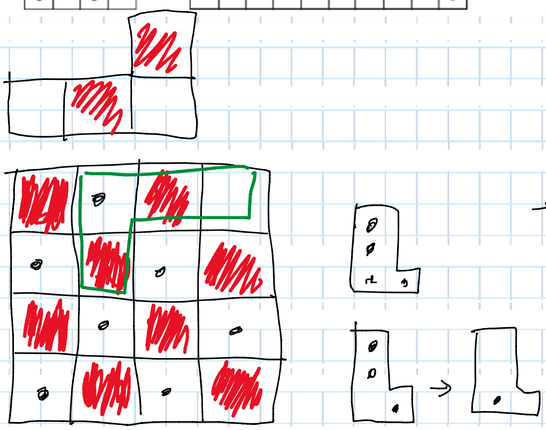






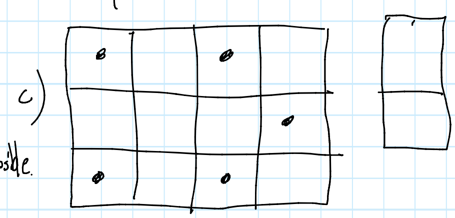


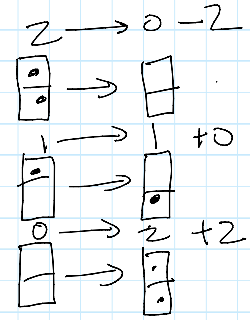
Parity of black dots is invariant



c)

Impossible





We start w/ 5 = odd dots
Can never get 0 = even dots

Tetromino always covers 2 white squares ⇒ Parity of the white dots is an invariant.

Originally 7 = odd, can't get 0 = even.

# original dots covered	4	3	2	1	0
# new dots afterwards	0	1	2	3	4
	−4	−2	+0	+2	+4

→ Parity of total dots invariant