Invariants I

BMC Int II Fall 2020

October 28, 2020

1 Colorings Warm Up

1. We want to use dominoes to tile an $m \times n$ board whose two diagonally opposite corner squares have been removed. Is this possible for a 6×8 board? A 5×7 board? What about a 6×7 board?

2. Can an 8×8 chessboard with the squares $C6$ and $G2$ cut out be tiled with dominoes?

3. Prove that a 10×10 board cannot be tiled with 1×4 tetrominos.

4. Can we cover a 4×5 rectangle with 5 tetris figures?

5. A class of 49 students have their desks arranged in a 7×7 square. No more than one student can fit in a single desk. The teacher asks every student to move to an adjacent desk (directly in front, behind, to the left, or to the right). Is it possible to fulfill the teacher's wish?

6. Can we tile an 8×8 board with 32 1×2 dominoes so that 17 of them are horizontal and 15 are vertical?

7. Prove that a 50×50 board cannot be tiled with T-shaped tetrominos.

8. A mouse is at a corner of a 5×7 board with its center square missing. Is it possible for the mouse to visit every other square on the board without visiting any square twice?

9. Now the mouse is at the corner of an 8×8 chessboard. Is it possible for the mouse to visit every other square on the board, without visiting any square twice, so that the mouse ends up at the opposite corner of the chessboard?

2 More Difficult Colorings

1. A $3 \times 3 \times 3$ watermelon cube and consists of 27 smaller cubical watermelon pieces arranged in a $3 \times 3 \times 3$ pattern. A caterpillar is eating the watermelon one small piece at a time and then moving to an adjacent piece. Is it possible for the caterpillar to start at one corner and

then finish at the central piece? (The watermelon is floating in water so the pieces will not fall down even if the pieces underneath are eaten)

2. Prove that a 4×11 rectangle cannot be tiled by L-shaped tetrominoes.

3. Can we tile an 8×8 chessboard with 1×3 trominos so that only one corner is uncovered?

4. We tile an 8×8 chessboard with 1×3 trominos so that one square is uncovered. What are the possibilities for this uncovered square?

5. Is it possible to tile a 13×13 board with 1×2 dominos only placed horizontally and 1×3 trominos only placed vertically?

6. Prove that we cannot tile a 6×6 square with 1×4 tetrominos.

7. A rectangular board was originally tiled by 1×4 tetrominos and 2×2 squares. All the pieces were then removed from the board but one 2×2 square was lost in the process. Instead, an additional 1×4 tetromino was provided. Prove that the board cannot be tiled with these new pieces.

8. A 7×9 rectangle is tiled with L-shaped trominos (3 squares) and 2×2 squares. How many 2×2 squares could we use?

9. We tile a 6×6 board with dominos. Prove that we can use a vertical or horizontal line to divide the board into two parts without cutting any domino.