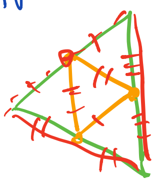
BMC Advanced 9/30

Rigid Motions, their geometry and algebra

I. Appetite wetters

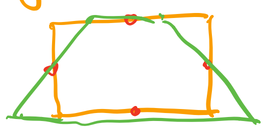
(1) How to construct a △ by midpoints of its sides? (or is it possible)



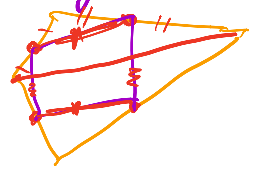
Yes!

(2) Same thing for a quadrilateral?

No?

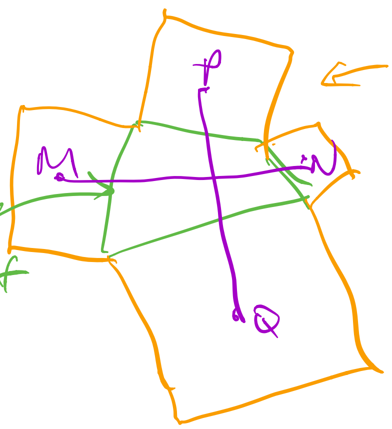


Moreover, for any quadrilateral, the midpoints of sides form a parallelogram



(3) what about a pentagon? Yes!

(4)

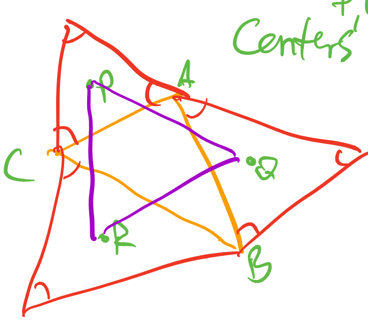


← squares

arbitrary quadri'lat

Claims: ① segments connecting centers of opposite squares are ⊥ and ② have equal length.

(5) "Napoleon's Theorem"



P, Q, R Centers of equilat. △'s constructed (outwards?) on sides of any △ form an equilateral △

II. Rigid motions (of the plane)

By a transformation we understand any mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$

a plane

e.g. translations, rotations, reflections

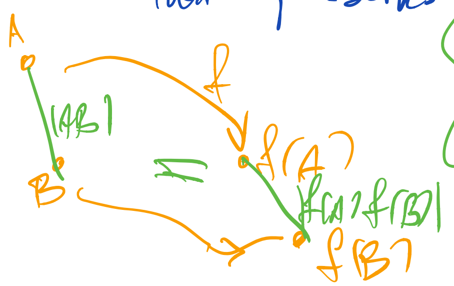
dialation $f(x,y) = (2x, 2y)$

projections e.g. $f(x,y) = (0,y)$

Def A rigid motion is a transformation that preserves distances, i.e. for all points A, B

$$|f(A), f(B)| = |AB|$$

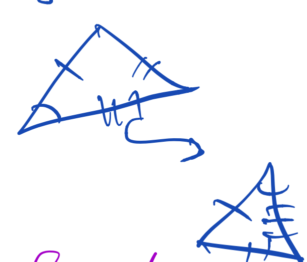
easy to believe that these transfs are rigid motions



Claim Any object on the plane is moved by a rigid motion to an equal (or congruent) object

Why? SSS tells this is so for triangles. Indeed it preserves all angles as well by SAS

by definition of congruence!



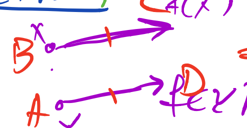
There are other classes of transformations (e.g. similarity (mult.) transfs, projective, circular, etc

For us this class is the "rigid motions" aka isometries

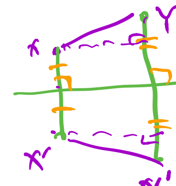
III. Classification of isometries

Claim 1 (a) Translations (b) Reflections (c) Rotations are isometries

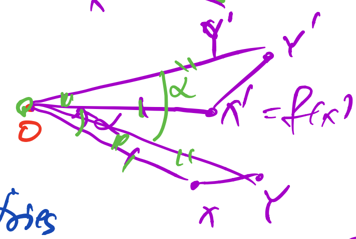
fix 0 pts
fix ⇒ 1 pts
fix 1 pt

(a) 

⇒ ABCD is a llgram ⇒ $|AB| = |CD|$

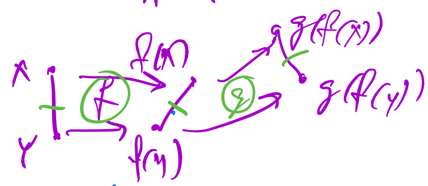
(b) 

⇒ $|XY| = |X'Y'|$

(c) 

by SAS $|XY| = |X'Y'|$

Claim 2 Composition of two isometries is an isometry



Claim 3 Inverse of an isometry is also an isometry

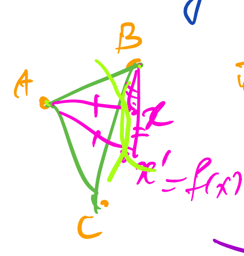
True, but not quite trivial, b.c. a priori we do not know that any isometry is invertible

(it can't be non-injective i.e. $x \neq y$ $f(x) \neq f(y)$ b.c. $|xy| \neq 0$ ⇒ $|f(x) f(y)| \neq 0$)

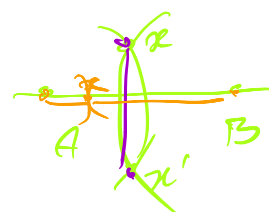
But why must it be onto?

e.g. for a right half plane  trans → sends it to a proper subset

Claim 4 An isometry w 3 non-collinear fixed pts is the identity transf.

i.e. 

if $f(A) = A$, $f(B) = B$, $f(C) = C$ then $f(x) = x$ for all x

⇒ 

$AB \perp xx'$ ← bisector

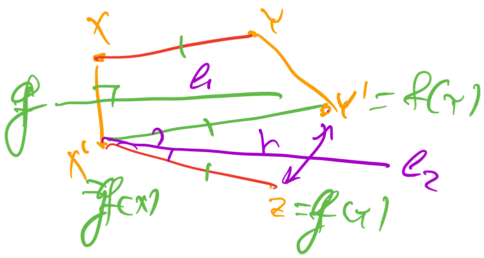
So will have AC and BC!

⇒ A, B, C are collinear →←

Claim 5 ✓ Any non-identity isometry with >1 fixed pt is a reflection
(was proved in proving claim 4)

Claim 6 Any isometry is a composition of at most 3 reflections

f isom $\quad f(x)=x'$, $f(y)=y'$

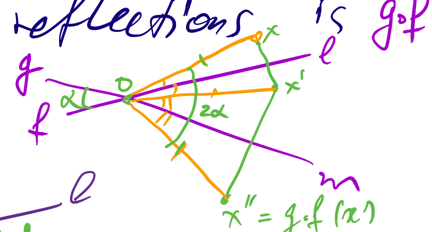


composition $h \circ g$
two reflections
sends x to x' and y to y'

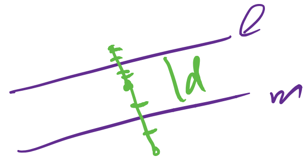
So by undoing them after f we'll get that $g \circ h \circ f$ fixes pts x & y
and by claim 5 this must be either Id or a reflection r

so f is either $h \circ g$ ← two reflections
or $h \circ g \circ r$ ← 3 reflections

Claim 7 Composition of 2 reflections is $g \circ f$
either a rotation by 2α if ℓ & m intersect



or translation in $\perp$ direction to $\ell \parallel m$ by $2 \times d$ btw them

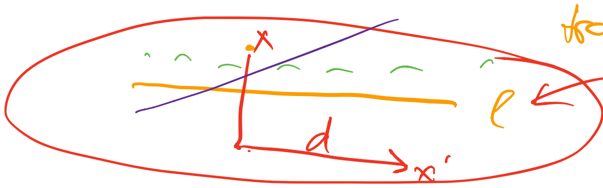


Claim 8

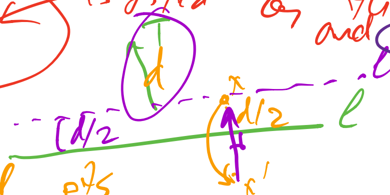
Composition of a reflection wrt line ℓ and rotation or translation is

either reflection (when translation$^{(a)}$ is $\perp$ to the line ℓ or when the $^{(b)}$ center of rotation is on ℓ)

or$^{(c)}$ glide reflection ~ composition of a reflection and translation $\parallel$ to the line



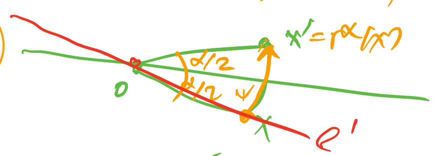
is fixed by this transf and only this one



Pf (a) look for fixed pts

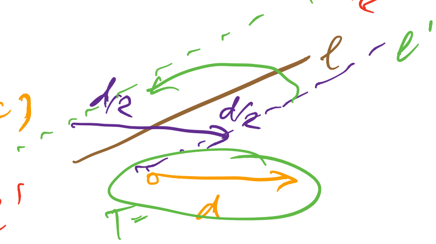
all pts of ℓ' - transl. of ℓ $\perp$ by $d/2$ are fixed, so this is refl. wrt ℓ'

(b)

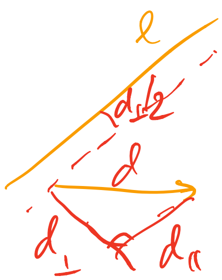


so x is fixed by the composition

(c)



$\Leftarrow$ need a new line ℓ' s.t. $\ell' \parallel \ell$ and is sent to itself by the composition



to do it convincingly factor the translation T into the composit. of two translation $T_\perp$ and $T_\parallel$

so $T = T_\perp \circ T_\parallel$ and so

$$R_\ell \circ T = R_\ell \circ T_\perp \circ T_\parallel = \overbrace{(R_\ell \circ T_\perp)}^{R_{\ell'}} \circ T_\parallel = \boxed{R_{\ell'} \circ T_\parallel}$$

glide reflection