I. Appetite wetters

(1) How to construct a \( \Delta \) by midpoints (or in impossible) of its sides?

(2) Same thing for a quadrilateral?

No?

Moreover, for any quadrilaterals, the midpoints of sides form a parallelogram.

(3) What about a pentagon? Yes!

(4) 

Claim: 1) Segments connecting centers of opposite squares are \( \perp \) and have equal length.

(5) "Napoleon's Theorem"

Centers of equilateral \( \triangle \)'s constructed on outward sides of any \( \triangle \) form an equilateral \( \triangle \).
II. Rigid Motions (of the plane)

By a transformation we understand any mapping \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) of a plane.

- Translations, rotations, reflections
- Dilations \( f(x,y) = (2x, 2y) \)
- Projections \( f(x,y) = (0, y) \)

Definition: A rigid motion is a transformation that preserves distance, i.e., for all points \( A, B \)

\[ |f(A), f(B)| = |AB| \]

It is easy to believe that these transformations are rigid motions.

Claim: Any object on the plane is moved by a rigid motion to an equal (or congruent) object.

Why? SSS tells this is so for triangles.

Indeed, it preserves all angles as well by SAS.

An definition of congruence!

There are other classes of transformations (e.g., similarity, projective, circular, etc.)

For us, this class is the "rigid motions" or isometries.
IV. Classification of Isometries

Claim 1
(a) Translations
(b) Reflections
(c) Rotations

Claim 2
Composition of two isometries is an isometry.

Claim 3
Inverse of an isometry is also an isometry.

Claim 4
A isometry \( \iff \) 3 non-collinear fixed pts.

\[ f(A) = A', f(B) = B', f(C) = C' \]

\[ \therefore f(x) = x \text{ for all } x \]

\[ \Rightarrow AB \perp B'C' \]

So will have \( AC \) and \( BC \).
Claim 5' Any non-identity isometry with \( r > 1 \) fixed \( f \) is a reflection (was proved in previous claim 4)

Claim 6 Any isometry is a composition of at most 3 reflections

\[
f(x) = x', \quad f(y) = y'
\]

The composition \( h \circ g \) of two reflections sends \( x \to x' \) and \( y \to y' \)

So by induction, the fixed points will be such that \( f \circ h \circ g \) fixes \( x \) and \( y \)

and by claim 5 this must be either \( \text{Id} \) or a reflection \( r \)

so \( f \) is either \( h \circ g \circ r \) or \( \text{Id} \) or \( 2 \) reflections

Claim 7 Composition of 2 reflections is \( g \circ f \)
either a reflection by \( 2 \theta \) or translated \( f \) or translation in \( \frac{1}{2} \) direction to \( \ell \) in \( \ell^{2} \) by \( 2 \cdot \theta \)
Claim 8: Composition of a reflection with line \( l \) and rotation or translation is either reflection (when translation is \( l \) to the line \( l \)) or glide reflection (for when the center of rotation is \( l \)).

Composition of a reflection and translation \( M \) to the line is fixed by this transformation. If \( a \) look for fixed points. All points of \( l \) translate to \( l \) \( \pm b/d \) and are fixed. So this is \( \pm b/d \) with \( l \).

So \( x \) is fixed by the transformation.

(c) Read a new line \( l' \) such that \( l'/l \) and \( l' \) is sent to itself by the composition. To do it convincingly factor the translation into the composition of two translations \( T \) and \( T'' \). So \( T = T_1 \cdot T_1 \) and so \( T_1 = T \cdot T_1 \) and so \( T_1 = (T \cdot T_1) \cdot T_1 = \frac{l}{l} \cdot T_1 \).