Combigeo Problem Solving

from the International Mathematical Olympiad Shortlists

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What follows are the combinatorial geometry IMO Shortlist problems since 1999 that are numbered at most 3.

Example 1. (2014 C1) Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

Example 2. (2003 C2) Let D_1, D_2, ..., D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs D_i.

Example 3. (2013 C2) A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

(i) No line passes through any point of the configuration.

(ii) No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Example 4. (2015 C2) We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

Example 5. (1999 G3) A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

Example 6. (2000 C3) Let $n \geq 4$ be a fixed positive integer. Given a set $S = \{P_1, P_2, \ldots, P_n\}$ of n points in the plane such that no three are collinear and no four concyclic, let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in their interior, and let

$$m(S) = a_1 + a_2 + \cdots + a_n.$$

Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

Example 7. (2003 C3) Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

Example 8. (2006 C3) Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

Example 9. (2011 C3) Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Example 10. (2016 A3) Find all positive integers n such that the following statement holds: Suppose real numbers a_1, a_2, $\ldots$, a_n, b_1, b_2, $\ldots$, b_n satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists ε_1, ε_2, $\ldots$, ε_n, each of which is either -1 or 1, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \leq 1.$$

Example 11. (2018 G3) A circle ω with radius 1 is given. A collection T of triangles is called good, if the following conditions hold:

1. each triangle from T is inscribed in ω;
2. no two triangles from T have a common interior point.

Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.