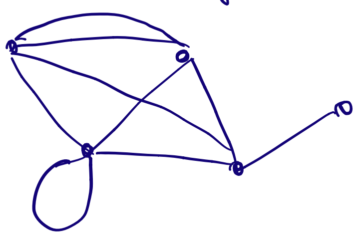
1. Planar graphs - review

Reminder: a <u>graph</u> G is a set of vertices V(G)
some of which are connected by edges
the set of edges is E(G)



A graph is <u>planar</u> if you can draw it in the plane without any edges crossing each other. Eg the graph above is planar:

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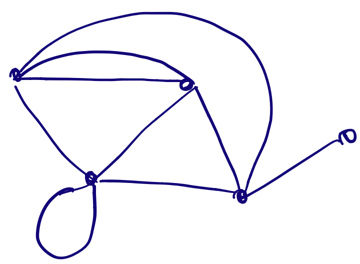
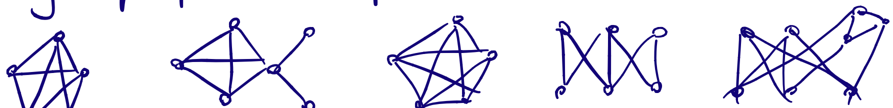


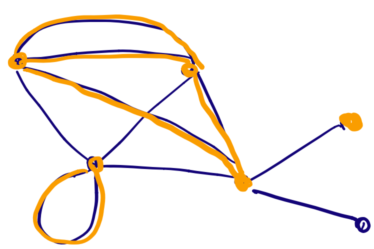
Q1: Which of the following graphs are planar? If not, how do you know?



Q2: Do you remember the theorem that helps us decide?

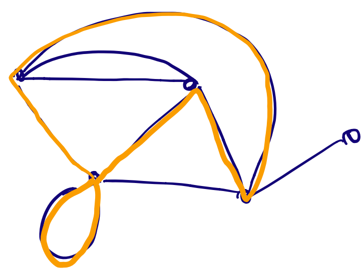
2. Some graph vocabulary

- A <u>subgraph</u> of a graph is subset of its vertices & a subset of its edges.
  which only touch
  the included vertices.

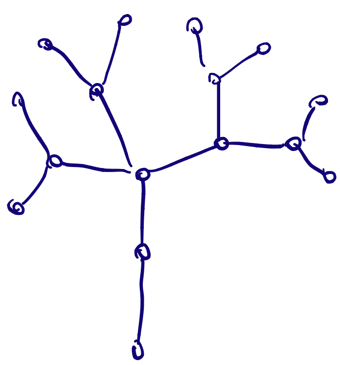


- A graph is <u>connected</u>, if you can reach any vertex from any other, walking along edges.
  ex: above

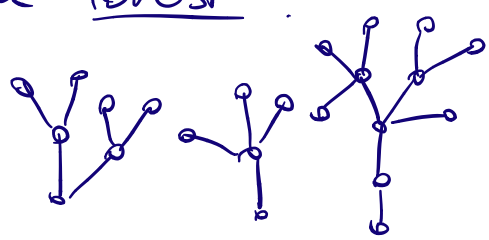
- A <u>cycle</u> is a walk in a graph along edges which gets back to its starting point.
  You can also think of it as a subgraph.



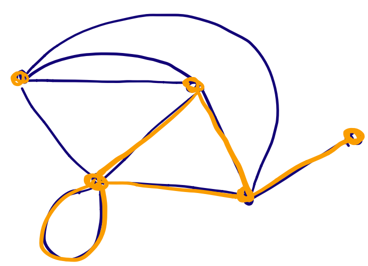
- A connected graph that has no cycles in it (other than walking somewhere & straight back) is called a <u>tree</u>.



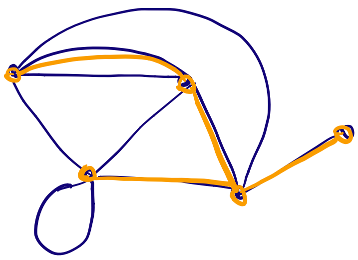
A disconnected cycle-free graph is a <u>forest</u>.



- A <u>spanning subgraph</u> of G is a subgraph that includes all of the vertices of G.



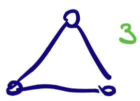
• If G is a connected graph, then a <u>spanning tree</u> is a spanning subgraph, which is a tree.



Q3: Prove that any tree with n vertices has $n-1$ edges.*

Q4: Find a spanning tree for each of the graphs below. Can you count how many spanning trees there are total?*

some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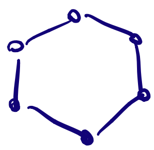
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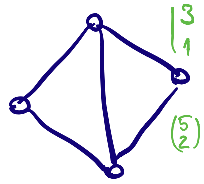
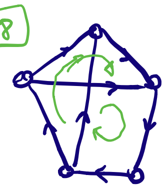

$\begin{vmatrix}3&1\\1&3\end{vmatrix}=\boxed{8}$

$\binom{5}{2}=10$



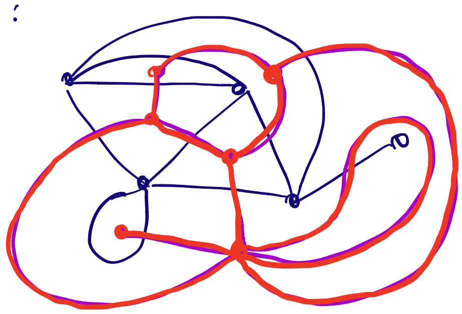
$\begin{vmatrix}3&1&2\\1&3&2\\2&2&5\end{vmatrix} = 3\overset{11}{\begin{vmatrix}3&2\\2&5\end{vmatrix}} - \overset{1}{\begin{vmatrix}1&2\\2&5\end{vmatrix}} + 2\overset{-4}{\begin{vmatrix}1&3\\2&2\end{vmatrix}} = 33-1-8=\boxed{24}$

Q5: A graph G has an edge which appears in every sp tree of G. What other property does such an edge have?

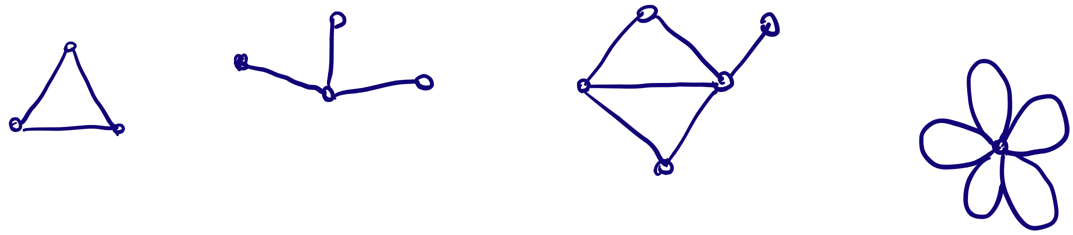
Q6: A <u>circuit</u> is a cycle that does not go through the same vertex. How many circuits are there in each of the graphs above?
(The starting pt of the circuit doesn't count, nor the direction.)

3. Planar graph duality

Each planar graph drawn in the plane has a <u>dual</u> planar graph G^*:

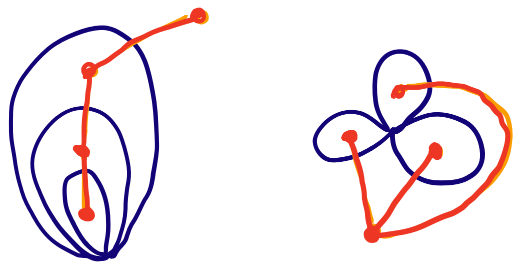


Q7: Draw the planar duals for the following planar graphs:



Q8: What kind of graph is the planar dual of a tree? Can you prove your answer?

Q9: Suppose you draw a planar graph in the plane two different ways. Will they have the same planar dual?



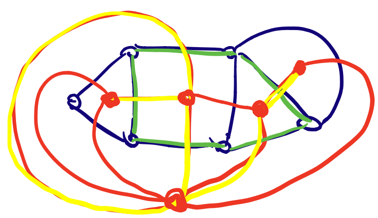
Q9.5 Suppose a connected graph G has an edge e so that removing e makes G fall apart to two components. What kind of edge is the dual e^* in the planar dual G^*?

Q10: Draw a planar graph and choose a spanning tree. Do the dual edges of the sp. tree give a sp tree for the dual graph? If not, what set of edges do? This is called the dual tree.

Q11: Draw a planar graph and choose a cycle. Look at the dual set of edges in the dual graph.

What property does this set of edges have? How could you describe it?

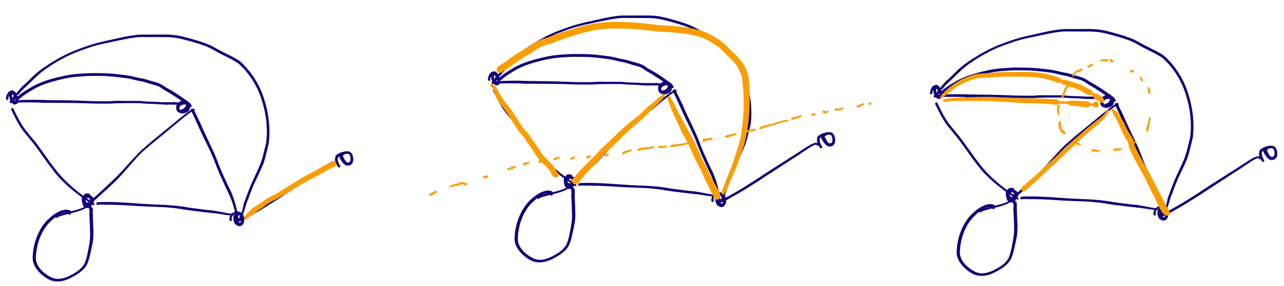
Ex:



For simplest examples, choose a circuit.

4. Cycles and cuts

A cut in a connected graph G is a set of edges C so that if you remove C, then G falls apart to multiple connected components



A minimal cut is a cut C so that if we add back any one edge of C, then G would once again be connected.

Q12 Prove that the planar dual of a circuit is a minimal cut.

Q13 Given G and a spanning tree T in G, there is a "fundamental cycle" corresponding to each edge of G that is not in T, and a "fundamental cut" corresponding to each edge of T. Can you guess what they are?

Q14 Prove that if G is a graph, T is a sp tree, G^* is the planar dual & T^* is the dual sp tree, then prove that fund. cycles for T are fund. cuts for T^*.

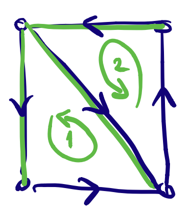
5. Some magic

Let G be any graph — not necessarily planar.

Sp trees, fund cycle, fund cut still makes sense.

Pick a direction for each edge

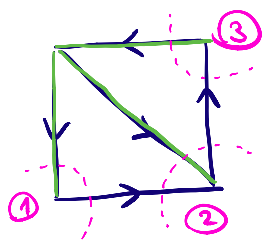
We'll make a matrix describing fund cycles & fund. cuts.



$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 3 & -1 \\ 2 & -1 & 3 \end{array}$$

count the # of shared edges between fund. cycles,
+ if same direction
− if opposite direction

count the length of the fund. cycle



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 1 \\ 3 & 0 & 1 & 2 \end{array}$$

# of coinciding edges
+ if same direction
− if opposite direction

# of edges in fund. cut

Miracle #1: these two matrices always have the same determinant!

$$\det\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8$$

$$\det\begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2\cdot\overbrace{\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}}^{5} + 1\overbrace{\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix}}^{-2} + 0\cdot\overbrace{\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}}^{0} = 10 - 2 = 8$$

<u>Miracle # 2</u>: This determinant is <u>always</u> = the # of spanning trees of G!

("The Matrix-Tree Theorem")