Rationals, Irrationals, and Continued Fractions

Online Berkeley Math Circle Intermediate II Ted Alper
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We can start by discussing some of the warmup questions. I do want to ask about this general question in particular: (warning: problem numbers may not be the same as in the pre-session handout!)

Problem 1 If I want to round $\sqrt{15} \approx 3.8729833462$), to the nearest tenth, how close can I come? Or the nearest hundredth? or to the nearest the nearest $1/N$ where N is any positive integer? Of course, if I round to the nearest ten-billionth, I can get all the decimal places I listed above. What might the advantages and disadvantages of that be?)

What ratio of integers M/N is closest to $\sqrt{15}$ if we insist that N be at most 10?

In general, do some values of N work better than others? On what basis should we decide which values of N work best?

What's a good Rational Approximation?

From the last problem, or otherwise... can we speculate on what might consitute a "good" rational approximation? And what questions about good rational approximations – however we define them – might we ask?

I don't think this one was even on the handout, but:

Problem 2 Suppose I want to plot points to draw a triangle that's as close to equilateral as possible, but need to use integer coordinates. Can I get an exact equilateral triangle? Why or why not? If not, how close can I come without having to use very large integers? (Maybe for simplicity we'll insist on a horizontal base).

Some other warmup problems

We can look at any of them if someone is interested, but I particularly want to touch on

Problem 3 How close to each other can two rational numbers with small denominators be? If a, b, c, d are positive integers with both $b \leq 1000$ and $d \leq 1000$ and $\frac{a}{b}$ and $\frac{c}{d}$ are *distinct* rational numbers (i.e. they aren't equal to each other), what is the closest they can be to each other? That is, what is the smallest possible value for

$$\left|\frac{a}{b} - \frac{c}{d}\right| ?$$

Whatever your answer, give both an example to show that there really are two such rational numbers that close to each other and give a justification why there can't be two such rationals any closer to each other than that.

Also these two

Problem 4 Purple Comet (2013) For positive integers m and n the decimal representation for the fraction $\frac{m}{n}$ begins 0.711 followed by other digits. Find the least possible value for n.

Problem 5 (British Math Olympiad #3 1987B) Find a pair of integers r and s such that $0 < s < 200$ and

$$\frac{45}{61} > \frac{r}{s} > \frac{59}{80}$$

and prove there is only one such pair r and s.

Some useful inequalities

Problem 6 (A sort of average of two fractions). If A, B, C, D are all positive, show that

$$\frac{A}{B} < \frac{A+C}{B+D} < \frac{C}{D}$$

We can give an intuitive argument or a rigorous one – or both!

Problem 7 Show that if

$$\frac{C}{D} - \frac{A}{B} = \frac{1}{BD}$$

(does that look familiar? and it's the same as $CB - AD = 1$), and E and F are positive integers for which

$$\frac{A}{B} < \frac{E}{F} < \frac{C}{D}$$

then $F \geq B + D$.

(A) How can we show this? (B) How often might this come up? (See BMO problem... also wait a little bit)

Dirichlet Approximation Theorem

Rational Approximations

Theorem . *(Dirichlet Approximation Theorem) For any positive real number* α*, and any positive integer* n*, there are positive integers* p *and* q *with* $q \leq n$ *for which*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qn} \leq \frac{1}{q^2}$$

How can we prove it? What does it suggest?

What questions?

What is a continued fraction?

compact notation for simple continued fractions:

$$[a_0, a_1, a_2, a_3, a_4] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}}$$

The inductive definition for continued fractions is that $[a_0] = a_0$, $[a_0, a_1] = a_0 + \frac{1}{a_1}$, and

$$\begin{aligned} [a_0, a_1, \ldots a_n, a_{n+1}] &= [a_0, a_1, \ldots, a_n + 1/a_{n+1}] \\ &= [a_0, a_1, \ldots, [a_n, a_{n+1}]] \end{aligned}$$

You should verify this yourself – it really is the same thing as the big fraction! And nothing so far depends on the a_k terms being integers, this is all just algebra. Of course, we can't divide by 0, but otherwise, everything works just like you'd expect.

You could also verify algebraically that $[a_0, a_1, \ldots a_n, a_{n+1}] = [a_0, [a_1, \ldots, a_n, a_{n+1}]]$. (This could be used to give a different inductive definition of continued fractions).

But how to compute them efficiently?

convergents of a continued fraction

It's helpful to define the **convergents** of a continued fraction

$$[a_0, a_1, \ldots, a_k] = \frac{p_k}{q_k}$$

so for example

$$[a_0] = \frac{p_0}{q_0} = \frac{a_0}{1}$$

and

$$[a_0, a_1] = \frac{p_0}{q_0} = \frac{a_0 a_1 + 1}{a_1}$$

But what comes after that?

$$[a_0, a_1, \ldots, a_k] = \frac{p_k}{q_k}$$

looking for patterns

Problem 8 Find the first six convergents for the continued fractions $[1,1,1,1,1,1]$, $[1,1,1,1,1,1]$, and $[2,1,2,1,2,1]$

n	a_n	fraction	p_n	q_n
0	1	1		
1	1	$1+\frac{1}{1}$		
2	1	$1+\frac{1}{1+\frac{1}{1}}$		
3	1	$1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}$		
4	1			
5	1			

n	a_n	fraction	p_n	q_n
0	2	2		
1	1	$2+\frac{1}{1}$		
2	2	$2+\frac{1}{1+\frac{1}{2}}$		
3	1	$2+\frac{1}{1+\frac{1}{2+\frac{1}{1}}}$		
4	2			
5	1			

Do we need another?

Problem 9 Evaluate $[1, 2, 3, 4, 5]$. After you do that, evaluate $[1, 2, 3, 4, 4]$ and $[1, 2, 3, 4, 6]$.

n	a_n	fraction	p_n	q_n
0	1	1		
1	2	$1+\frac{1}{2}$		
2	3	$1+\frac{1}{2+\frac{1}{3}}$		
3	4	$1+\frac{1}{2+\frac{1}{3+\frac{1}{4}}}$		
4	5			

Something's going on! Can we prove it algebraically?

We can define the *convergents* of the continued fraction by

$$\begin{array}{rclcrcl} p_0 & = & a_0, & \text{and} & q_0 & = & 1 \\ p_1 & = & a_0 a_1 + 1, & \text{and} & q_1 & = & a_1 \\ p_{n+1} & = & a_{n+1} p_n + p_{n-1}, & \text{and} & q_{n+1} & = & a_{n+1} q_n + q_{n-1} \end{array}$$

(the last line is for $n \geq 1$, of course)

And we can then prove by induction on k that, for any finite continued fraction,

$$[a_0, a_1, \ldots, a_k] = \frac{p_k}{q_k}$$

Next we'll show, for any k, $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ which means

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

(The proof is also by induction on k). While *none* of the proofs thus far require the a_k terms to be integers, What does this tell us when they *are* integers? (Go back to the warmup problem) successive convergents of a continued fraction are always as close to each other as rational numbers are allowed to be.

connection to Euclidean algorithm

Problem 10 Find the simple continued fractions expansion for $81/35$ and $277/101$ and use these to find integers m and n for which $81m + 35n = 1$ and $277m + 101n = 1$

Problem 11 Find the continued fraction expansions for $\dfrac{10! + 1}{11! + 1}$ and $\dfrac{3^7 - 1}{3^8 - 1}$.

Convergence of infinite simple continued fractions

- infinite simple continued fractions (with integer coefficients) always converge
- For any positive real number, there is a (almost unique) continued fraction (with integer coefficients) that converges to it

Problem 12 Evaluate the real number to which the following continued fraction converges:

$$[5, 2, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, \ldots]$$

(I want an answer with a square root in it, not a decimal).

Every infinite continued fraction converges to an irrational number; Every irrational number has an infinite continued fraction expansion that converges to it.

Every periodic (repeating) continued fraction converges to an irrational root of a quadratic equation. Also every irrational root of a quadratic equation has a continued fraction expansion that is (eventually) periodic.

Problem 13 Find the continued fraction expansion for $\sqrt{20}$

how good a rational approximation is the convergent of a continued fraction?

Continued fractions and Pell's equation of form

$$x^2 - dy^2 = \pm 1$$

We can also apply continued fractions to Pell's equation $x^2 - dy^2 = 1$. (d is NOT a perfect square here) In particular, we can show that all solutions for x and y in positive integers come from the convergents of continued fractions for $\sqrt{d}$, where $x = p_n$ and $y = q_n$.

In fact, for every convergent p_n, q_n for $\sqrt{d}$, we must have $|p_n^2 - dq_n^2| \leq \lceil 2\sqrt{d} \rceil$, where $\lceil x \rceil$ stands for "*the least integer greater than or equal to* x".

And if you have any two solutions, $a^2 - b^2\sqrt{d} = u^2 - v^2\sqrt{d}$, then if you multiply out $(a + b\sqrt{d})(u + v\sqrt{d}) = x + y\sqrt{d}$, i.e. $x = au + bvd$, $y = av + bu$, you can verify that $x^2 - y^2 = 1$.

Problem 14 Find 2 solutions in positive integers for $x^2 - 6y^2 = 1$.

Upshot

Theorem . *If* α *is any irrational number and* p_n,q_n *and* p_{n+1}, q_{n+1} *are consecutive convergents of the continued fraction approximation for* α*, then*

(i) *if* n *is even,* $\frac{p_n}{q_n} < \alpha < \frac{p_{n+1}}{q_{n+1}}$*, if* n *is odd,* $\frac{p_{n+1}}{q_{n+1}} < \alpha < \frac{p_n}{q_n}$.

(ii) *For all* n*,* $|\alpha - \frac{p_n}{q_n}| < \frac{1}{(q_n)^2}$ *(could even improve this somewhat!)*

(iii) *For any integer* $q < q_n$ *and any integer* p*,* $|\alpha - \frac{p_n}{q_n}| < |\alpha - \frac{p}{q}|$

This could be restated as: **(i)** the convergents alternate between being greater than and less than the number they are approximating, **(ii)** they are *very* good rational approximations and **(iii)** they are the best rational approximations of any rational approximations with the same or smaller denominator.

Some deeper results

Theorem . *(Liouville, 1844) If $f(x)$ is an nth degree polynomial with integer coefficients, and α is a real root of that polynomial, then there is a number $\kappa > 0$ such that for all rational numbers $\frac{p}{q} \neq \alpha$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\kappa}{q^n}$$

One consequence of this theorem: transcendental numbers exist! (of course, that's a consequence of Cantor's theorem as algebraic numbers are countable; but that came 30 years later (c. 1874) – Lindemann proved pi was transcendental in 1882.)

Theorem . *(Thue-Siegel-Roth, 1909, 1921, 1955) If $f(x)$ is an nth degree polynomial with integer coefficients, and α is a real root of that polynomial, then there is a number $\kappa > 0$ such that for all rational numbers $\frac{p}{q} \neq \alpha$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\kappa}{q^{2+\epsilon}}$$