BMC Intermediate II, January 2020
Six truths and a lie (?)

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Today we will be building up to something called an *error-correcting code*, which allows us to deduce some information even when a mistake has been made (or a lie has been told).

1. List the binary expressions for the integers 1 through 7. We'll need them later.

Number	Binary Expression
1	
2	
3	
4	
5	
6	
7	

2. Create the arithmetic tables for addition and multiplication mod 2.

+	0	1
0		
1		

*	0	1
0		
1		

3. Let's have a quick primer on how to multiply matrices.

(a) A small one first. We multiply "row by column".

$$\begin{bmatrix} 1 & 2 & 8 \end{bmatrix} \begin{bmatrix} 21 & 0 \\ 3 & -4 \\ -2 & -1 \end{bmatrix} =$$

(b) Next we'll go bigger and do our arithmetic mod 2.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} =$$

4. Did you notice anything unusual about the second matrix in the problem we just did? Look back at previous questions if you need a hint.

5. Let's recap a short version of my old birthday mind-reading trick. Instead of figuring out birthdays, we'll have a person select an integer between 0 and 15, inclusive. For this, we will only need four "cards" or questions. Remember, this mathemagic trick is related to binary numbers. List the cards below and write a quick description of how to do the trick.

Test out a few numbers with your partner. With four yes/no questions, we should be able to distinguish between 16 different answers, as $16 = 2^4$. No lying yet!

6. Now we get fancy. We are going to ask our partners to secretly select an integer between 0 and 15 again. This time we will ask SEVEN questions, but the partner who chose the number is allowed to lie when answering ONE of the questions if they want. (Or they can answer all truthfully.) Our goal is to figure out what number they selected.

I will demonstrate first, and then I will show you some calculations that will allow you to figure out the trick as well.

Instructions
Think of a number between 0 and 15.
  Now answer the following questions.
  *You are allowed to lie **once**.*

  1. Is the number 8 or greater?
  2. Is it in the set $\{4, 5, 6, 7, 12, 13, 14, 15\}$?
  3. Is it in the set $\{2, 3, 6, 7, 10, 11, 14, 15\}$?
  4. Is it odd?
  5. Is it in the set $\{1, 2, 4, 7, 9, 10, 12, 15\}$?
  6. Is it in the set $\{1, 2, 5, 6, 8, 11, 12, 15\}$?
  7. Is it in the set $\{1, 3, 4, 6, 8, 10, 13, 15\}$?

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7. Select an integer between 0 and 15 and record it here. (No need to be secret right now, as you and your partner are doing some behind the scenes work to figure out the trick.)

   Starting number: _____

   Now, fill in the following table for each of the lying (or not) options. The first column indicated which question you lie about. In the second column, write the 1×7 answer matrix A you get when answering the questions. In the third column, compute the (mod 2) matrix product AB, where B was the notable 7×3 matrix of binary expansions in question 3b. You will probably need some scratch paper for the matrix multiplications.

Lie	A	AB
none		
1		
2		
3		
4		
5		
6		
7		

8. Compare to a team who tried a different number. Are there any similarities in your answers? Do you have a conjecture for how to determine the number? Once you think you have a plan, try it out with secret selections with your partner.