How many points are on a plane? II.

Recall from our last meeting

DEFINITION A PLANE is a set, with elements called POINTS, with special subsets called LINES which satisfy the following properties.

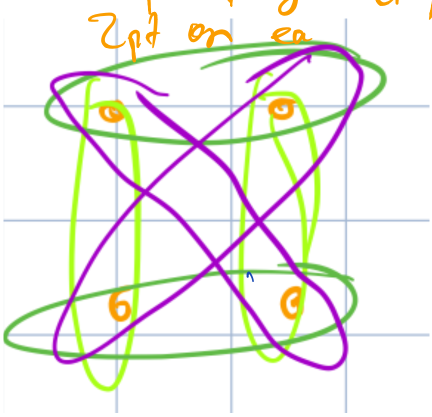
A1. (Incidence Axiom) Every two *distinct* points belong to a *unique* line.

A2. (Paralles Axiom) For every line and a point *not on it* there is a *unique* line containing this point parallel to the given line.

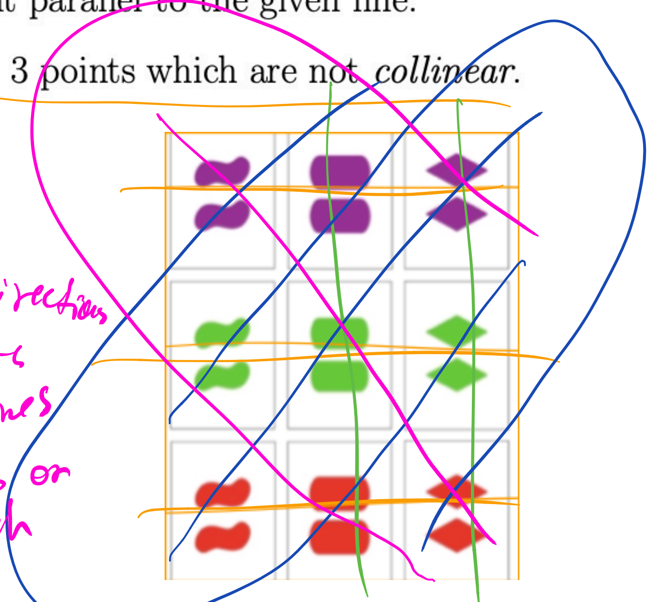
A3. (Dimension Axiom) There exist 3 points which are not *collinear*.

4 point geometry
2pt on ea

6 lines



4 directions
9 pts
12 lines
3 pts on each



Theorem 1. Every line has at least two points.

Theorem 2. Every two lines on a plane have the same number of points.

Theorem 3. There are at least 4 points on a plane.

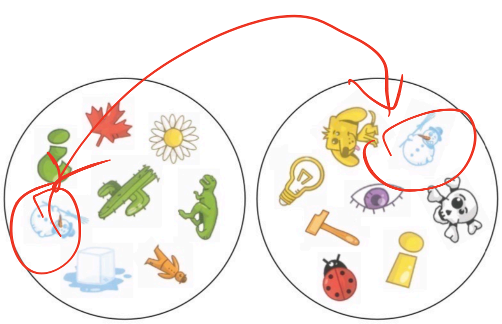
Theorem 4. There exist planes with 4 or 9 points.

yes no yes No ??? no only n^2

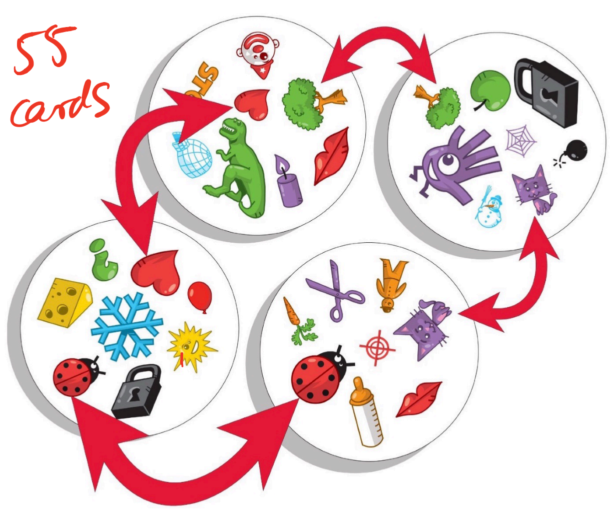
Can 16, 25, 36, 49, 64, 10^2, 12^2 points occur

Questions:

- Are there planes ~~with 5,~~ ... 8 points?
- How many points *can* there be ?
- How many *lines* are on a plane?
- What about the *other game*?
- Is there a *3D (4D, etc) version*?



55 cards



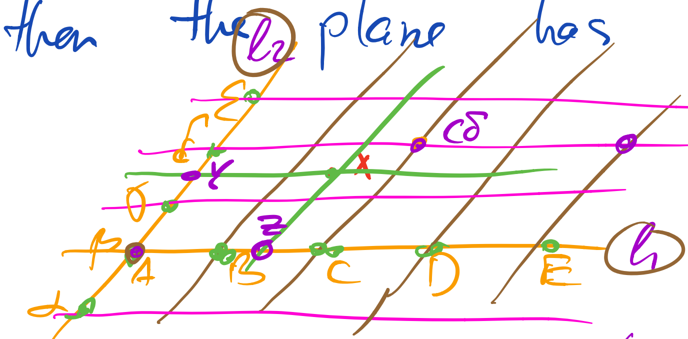
Answers

How many lines are on this plane? $n+1$ through each pt so $\frac{n^2(n+1)}{n} = \boxed{n(n+1)}$

Th 5 If a line has n points, then the plane has n^2 points

Lines

Proof



Draw all lines parallel to ℓ_1 and ℓ_2 and count their intersections

This will give n^2 points (think coordinates)

and no more points will be on the plane b.c. lines through X ∥ to ℓ_1 (or ℓ_2) will intersect ℓ_2 (or ℓ_1) and so X will happen to be one of already obtained pts.

So there are n^2 pts total //

Q2 Can we have n^2 points with $n = 4, 5, 6$?

More generally, for which n there's a plane with n^2 pts

Partial answer for good n's (complete answer for dim 3 and more)

$n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19,$

$23, 25, 27, 29, 31, 32, 37, 41$

all primes and prime powers

$n = p^k, k \geq 1$
p - prime

Will see how this works for $n = p$

Graeco-Latin squares

Greek letters

Roman (Latin?) letters

Aα	Bγ	Cβ
Bβ	Cα	Aγ
Cγ	Aβ	Bα

γ

α

β

Aα	Bδ	Cβ	Dε	Eγ
Bβ	Cε	Dγ	Eα	Aδ
Cγ	Dα	Eδ	Aβ	Bε
Dδ	Eβ	Aε	Bγ	Cα
Eε	Aγ	Bα	Cδ	Dβ

Euler in 1781 asked for which n a GL $n \times n$ square exists

Aα	Bγ	Cδ	Dβ
Bβ	Aδ	Dγ	Cα
Cγ	Dα	Aβ	Bδ
Dδ	Cβ	Bα	Aγ

Conjectured that there are none for $n=6$

(Proved in 1901)

There's a very close connection between these GL squares and our planes

Th In fact each geometry with n pts on a line gives rise to $n-1$ "orthogonal" Latin squares

$$\left(\begin{array}{|ccc|} \hline A & B & C \\ B & C & A \\ C & A & B \\ \hline \end{array} , \begin{array}{|ccc|} \hline A & C & B \\ B & A & C \\ C & B & A \\ \hline \end{array} \right) = \begin{array}{|ccc|} \hline AA & BC & CB \\ BB & CA & AC \\ CC & AB & BA \\ \hline \end{array}$$

orthogonal L. squares

all 9 pairs are different (i.e. a GL square)

So no geometry w 36 points (b.c. no GL-square)

Constructing geometries w p^2 points (prime)

Descartes "defined" lines on "usual" plane as solutions of linear equations in 2 variables $\{(x,y) \mid ax+by=c\} = l_{abc}$

$y = kx+b$ ← not all lines

or $x = c$

or together we can write as

$$ax + by = c \quad (\text{where } (a,b) \neq (0,0))$$

algebraic def of lines and it works

i.e. A1, A2, A3 can be easily verified e.g. to draw line through $A(x_1, y_1)$, $B(x_2, y_2)$ we solve for k, b in $x_1 \neq x_2$

$$\begin{cases} y_1 = kx_1 + b \\ y_2 = kx_2 + b \end{cases} \quad \Bigg| \quad y_1 - y_2 = k(x_1 - x_2)$$

$$\Rightarrow k = \frac{y_1 - y_2}{x_1 - x_2}, \quad b = \ldots$$

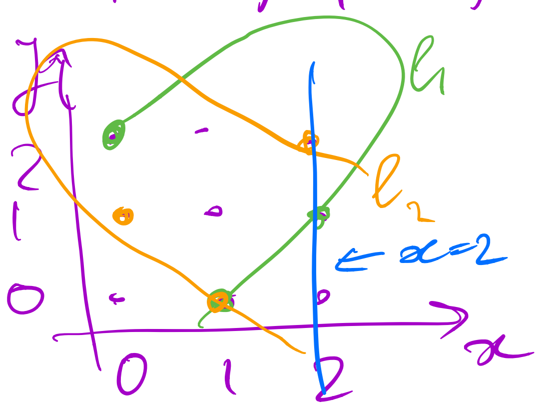
$\Rightarrow$ unique line through A, B

To get a plane with p^2 points, p prime, take for a "line" the set of remainders mod p, i.e. ("coordinates")

$L = \mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$

So our p-plane is the set

$P = \{(x,y) \mid x, y \in L\}$ e.g. for $p=3$



Lines = solutions of lin. equations (mod p)

$ax + by = c$

i.e. if $b \neq 0$ mod p

then $y = kx + d$

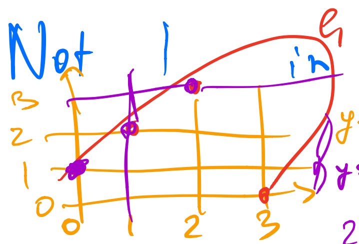
$k, d \in \mathbb{Z}_p$

e.g. $y = x+2$ gives l_1

$y = 2x + 1$ gives l_2

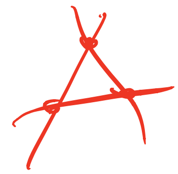
Axioms work b/c we can divide in $\mathbb{Z}_p$ by $\neq 0$ elements

Not in non-prime case e.g. mod 4 $\leftarrow$ no division in $\mathbb{Z}_4 = \{0,1,2,3\}$



Fixing inequity between points and lines

PROJECTIVE GEOMETRY



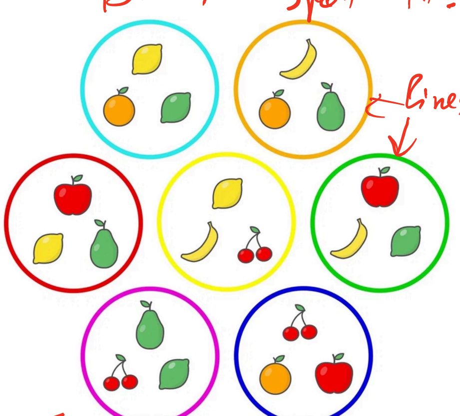
DEFINITION A PROJECTIVE PLANE is a set (of POINTS) with a collection of subsets (called LINES) satisfying the following AXIOMS.

P1. (Incidence Axiom) Every two distinct points belong to a unique line.

P2. (No Paralles Axiom) Every two distinct lines have a common point.

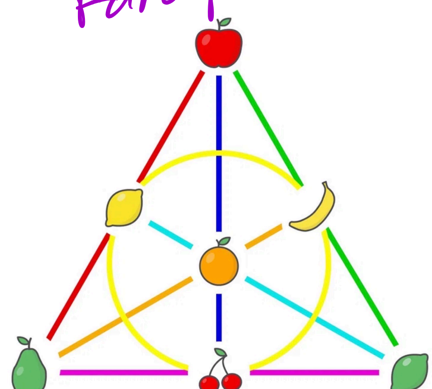
P3. (Dimension Axiom) There are 4 points no three of which are collinear.

Baby Spot it!



7 fruit proj. plane

Faro plane



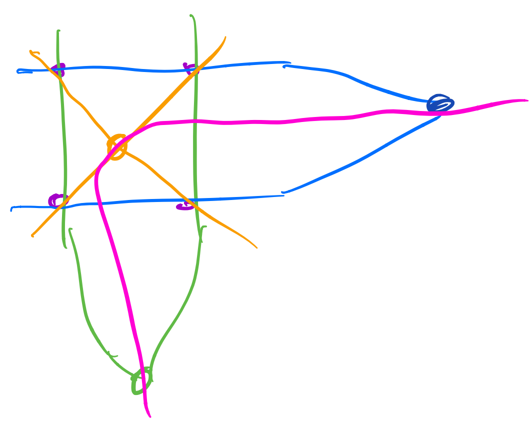
7 pts, 7 lines

A proj plane w 3 lines/point 3 pts per line

Thm Every projective plane comes from a "usual" plane by adding a new "∞" line intersecting each group of // lines in a unique point

So if there are $n+1$ pts on a line, there are n^2+n+1 pts on this plane, and same n^2+n+1 lines

Ex



= Fano plane

Conversely removing any line from a projective plane (w n^2+n+1 pts) gives a "usual" plane w n^2 pts