BMC 2020 Advanced
Darts and Balls - A Game to Always Win

I. Warm-Up/Pre-reading

1. What is the volume of a sphere (technically a 3D ball)? Can you derive the formula (probably from calculus)?

2. If you do not know calculus, here is a brief picture to explain:
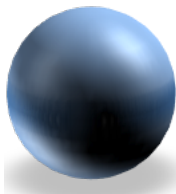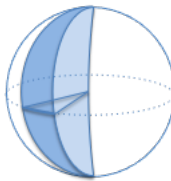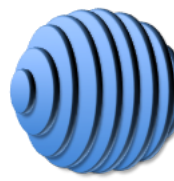

Circle   Rings   Slices   Boards

Sphere   Shells   Wedges   Plates

   a) In the picture above, the Circle has radius 4. What is its area?

   b) The picture describes a few ways to approximate the area of the circle (if we don't already know the formula) by cutting the circle into pieces whose area we know how to calculate. Suppose we cut the circle into 8 *Boards*, starting from the right, what is approximated area of the circle?

   c) The idea of calculus is to increase the amount of boards (rectangles) to get better approximation, and the limit of this approximation as we increase the number of boards to infinity will be the actual area of the circle.

   d) Repeat this process to find volume of the sphere: You should spend some time thinking about cutting a sphere into pieces (we call them discs - the picture calls them plates). What shape is each disc in? What is the volume of each disc?

II. A Dart Game

3. Some warm-ups (AMC-8 style):

   a) In a dice game, Harry wins \$3 for each throw that lands on 5 or higher and lose \$2 otherwise. What is Harry's average earning/loss per throw?

   b) What is the probability that when you randomly pick 2 real numbers between -1 and 1, the sum of their squares is at most 1?

   c) Harry hangs his dartboard on a circumscribing square background (as you see on the wall). Assuming that Harry is good enough to always hit the square and terrible enough to have no aim, what are the chances that Harry hits the dartboard?

4. Harry is training for better aim to join a dart league. So he sets up a game with two motivations:

- The game should get harder as it goes.
- He gets rewarded for hitting close to the bulls-eye.

So let's agree on the following game:

- To encourage Harry to hit the bullseye, at the first throw, we are going to make the entire dartboard the bullseye.
- For the next throw, we will shrink the bullseye by making the new radius half the perpendicular chord through the previous hit.
- We continue shrinking after every hit. The game ends when Harry misses.
- For every hit, Harry gets a dollar from each student in BMC Advanced.

> The major question for today is: How much does Harry make per game?

III. Balls of high-dimension

Some important players:

5. The **Gamma Function**: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. People say Gamma Function is the extension of the Factorial Function. Why?

6. The **Beta Function**: $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$. Are Gamma and Beta related?

7. Free fact (or you can prove it yourself): $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

8. What is the volume of the $2n$-dimensional unit ball?

9. (Take-home) What is its surface area of the $2n-$dimensional unit sphere?