25 Point Geometry I

BMC Advanced Fall 2020

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1 Introduction

We will consider a finite form of geometry which is similar to what we know in Euclidean geometry but not exactly the same. Thank you to Thomas Kilkelly for allowing me to use his problems and for the idea.

Definition 1.1. A **point** is a letter A through Y and a **line** is a row or column in one of the three blocks below:

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

A	I	L	T	W
S	V	E	H	K
G	O	R	U	D
Y	C	F	N	Q
M	P	X	B	J

A	X	Q	O	H
R	K	I	B	Y
J	C	U	S	L
V	T	M	F	D
N	G	E	W	P

Remark. Note that there are a total of 30 lines and each line contains 5 distinct points. Also, each point is on six distinct lines.

Example 1.2. Given any two distinct points, there exists a unique line that contains both of them.

Definition 1.3. We say two lines are parallel if they have no points in common.

Exercise 1.4. Convince yourself that parallel lines are two rows or columns in the same block.

Example 1.5. For any two distinct lines, they are either parallel or they have only one point in common.

Example 1.6. Given a line and a point not on the line, there is a unique line containing the point that is parallel to the given line.

Definition 1.7. Two lines are **perpendicular** if one of them is a row and the other is a column in the same block.

Definition 1.8. The **length** of a line segment is the shortest separation of the two endpoints in the row or column, where they are thought of as cyclic. Row lengths and column lengths are distinct.

Example 1.9. The distance between A and E is just one row-unit and the distance between A and V is two column-units. There are five different lengths: 0, 1 row-unit, 2 row-units, 1 column-unit, and 2 column-units.

Definition 1.10. We say that r is the midpoint of p and q if p, q, r all lie on the same line and the distance between p and r is the same as the distance between r and q.

Exercise 1.11. How are the second and third blocks created from the first block?

2 Polygons

Definition 2.1. A **triangle** is a set of three non-collinear points.

Exercise 2.2. How many triangles are there?

Exercise 2.3. Find an example of an equilateral, isoceles, and scalene triangle.

Exercise 2.4. Can we find a scalene triangle that is not a right triangle?

Exercise 2.5. Prove that the circumcenter of a right triangle is the midpoint of the hypotenuse.

Exercise 2.6. Verify that the circumcenter exists for any triangle. Can you prove it?

Exercise 2.7. Verify that the centroid exists for any triangle. Can you prove it?

Definition 2.8. We give a limited way to compare lengths.

Exercise 2.9. Show for several triangles that the centroid divides the medians in ratio $2:1$.

Exercise 2.10. Pick several triangles and show for them that the Euler line exists; the circumcenter, centroid, and orthocenter all collinear.

Definition 2.11. A **quadrilateral** is a set of four points so that no three are collinear.

Exercise 2.12. How many different quadrilaterals are there?

Exercise 2.13. Are there any squares?

Example 2.14. There are 10 different types of quadrilaterals:

- rectangle
- rhombus

- general parallelogram
- right trapezoid with 3 congruent sides
- right trapezoid with 2 congruent sides
- general right trapezoid
- isoceles trapezoid
- kite
- dart
- general quadrilateral

Exercise 2.15. Find an example of each type of quadrilateral.

Exercise 2.16. If the opposite sides of a quadrilateral are parallel and congruent, then is the quadrilateral a parallelogram?

3 Conics

Definition 3.1. A **circle** is the set of points a given distance away from a given point.

Exercise 3.2. How many points are on a circle?

Exercise 3.3. How many circles are there?

Definition 3.4. A **tangent** to a circle is a line that intersects the circle at a unique point.

Exercise 3.5. Is it true that a tangent to a circle is perpendicular to the radii to the intersection point?

Exercise 3.6. The nine-point circle is the circle that contains the midpoint of each side of the triangle, the food of each altitude, and the midpoint of the line segment from each vertex of the triangle to the orthocenter. Draw several triangles and show that this nine-point circle exists for this geometry.

Exercise 3.7. Does a line through the center of a circle have to always intersect the circle at two points?

Definition 3.8. A **parabola** is a set of points equidistant from a given line and a given point not on that line.

Exercise 3.9. How many points are on a parabola?

Exercise 3.10. How many parabolas are there?

Definition 3.11. A **tangent** to the parabola intersects the parabola at a unique point and is not perpendicular to the directrix. The **vertex** is the point on the parabola which lies on the altitude from the focus to the directrix.

Exercise 3.12. Is it still true to the tangent to a parabola at the vertex is parallel to the directrix?

Exercise 3.13. What can we say about tangent lines to the endpoints of a focal chord?