BMC INTERMEDIATE II: CYCLES, CUTS AND PLANAR DUALITY

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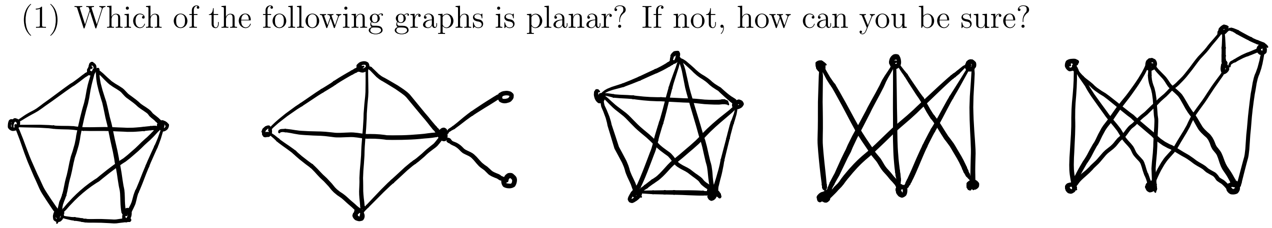
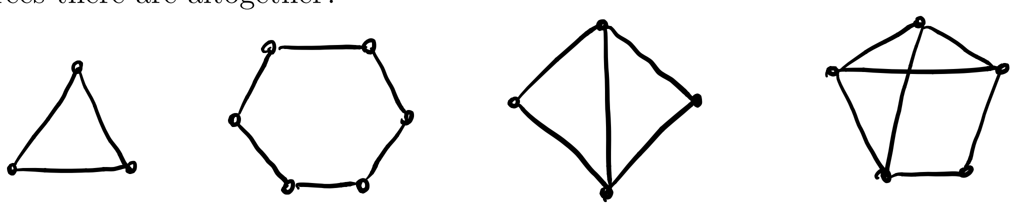
1. Planar Graphs and Vocabulary

(1) Which of the following graphs is planar? If not, how can you be sure?



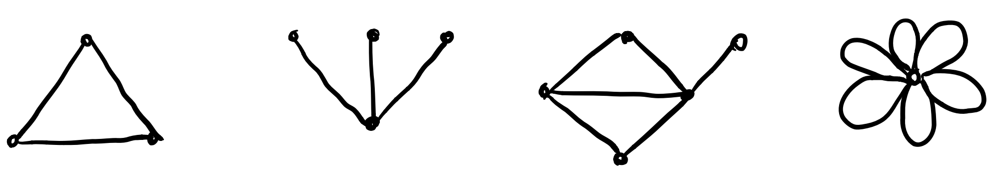
(2) Do you remember the theorem that helps us decide whether a graph is planar?
(3) Prove that any tree with n vertices always has exactly $n-1$ edges.
(4) Find a spanning tree for each of the graphs below. Can you count how many spanning trees there are altogether?



(5) A graph G has an edge which appears in every spanning tree of G. What other property must such an edge have?
(6) A *circuit* is a cycle that does not go through the same vertex twice. How many circuits are there in each of the graphs of Question 4? (Here the starting point of the circuit doesn't matter, nor the direction, only the set of edges.)

2. Planar Graph Duality

(7) Draw the planar dual for the following planar graphs:

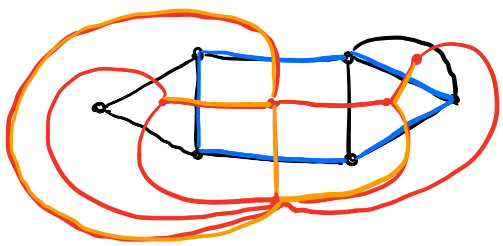


(8) What kind of graph is the planar dual of a tree? Can you prove your answer?
(9) Suppose you draw a planar graph in the plane two different ways. Will they have the same planar dual?

(10) A graph G has an edge e so that removing e makes G fall apart to two connected components. (Such an edge is called a *bridge*.) What kind of edge is the dual e^* in the planar dual G^*?

(11) Draw a planar graph and choose a spanning tree, Do the dual edges to the spanning tree edges give a spanning tree of the dual? If not, what (related) set of edges do give a dual spanning tree? Can you prove that this is always the case?

(12) Draw a planar graph and choose a cycle. Look at the dual set of edges in the dual graph. What property does this edge set have? How could you describe it? Hint: for suggestive examples choose a circuit, like the one below.



3. Cycles and Cuts

(13) Prove that the planar dual of a circuit is a minimal cut.

(14) Given G and a spanning tree T in G, there is a *fundamental cycle* corresponding to each edge of G that is not in T, and a *fundamental cut* corresponding to each edge of T. Can you guess what these are?

(15) Prove that if G is a planar graph, T is a spanning tree, G^* is the planar dual and T^* is the dual spanning tree, then prove that fundamental cycles for T are dual to fundamental cuts for T^*.

(16) For each graph in Problem (4), choose an orientation for the edges, choose a spanning tree, and write down the cut and flow matrices. Compute their determinants. Do the determinants depend on the choice of spanning tree? On the choice of orientation? Do you notice anything about the value of the determinants?

4. Bigger Questions to Think About

(1) What about duality for graphs that are not planar? (For some food for thought, you can look into: matroids, cycle matroids of graphs, matroid duality, hyperplane arrangements, Gale duality.)

(2) How might one prove the Matrix-Tree theorem? (You need some knowledge about linear algebra: vectors, matrices, bases, determinants.)