BMC - Intermediate II: Undergraduate Algebraic Topopology ("Alg Top")

200930 - Chris Overton (revised after class on 200101)

Caution: these notes are not written to be fully understandable to you on their own, although they are filled in a bit from yesterday's class presentation. We tried to cover details in class, and you are encouraged to look up terms!

Alg Top is one of the major areas of math. You use it to map from topological spaces to objects you can study using modern algebra.

There are surprisingly few Algo Top texts aimed at undergrads - most are for graduate students, maybe allowing in some "advanced undergraduates."

Why? Because you first need to know some things about 1) Algebra, 2) Topology, and then you combine these in new ways.

We will skimp a bit on both of these in order to allow more time for Alg Top. Think of this as sort of a mathematical "taste test."

Our presentation is influenced by the following texts, but our two hours also need to introduce some prerequisites to these.

- Greenberg & Harper (quickest tour that focuses on todays topics)
- Massey
- Munkres (friendly introduction)
- Hatcher (the current "standard" in graduate texts)
- Bredon's *Topology and Geometry* (excellent content, but plenty of it not Alg Top.)

There is also a good modern introduction by Burt Totaro of UCLA in *The Princeton Companion to Mathematics* (This is just about the only of his dozens of papers I can't find online.)
This talk is inspired by my attempt to introduce knot theory (one Alg Top area) to BMC Advanced this Summer, but over four hours.

Today's topics

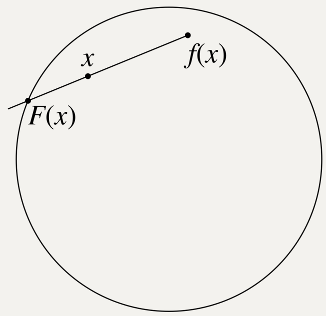
- In the forest of math, which trees do you climb? We'll start by looking at two of them: (modern) algebra and (point-set) topology.
- Then we'll gain exposure to **functors** from topological to algebraic **categories**. In English: you'll learn about ways to map from 'sets' of topological spaces into 'sets' of **algebraic** objects in ways that let you use algebra to study the spaces.
- In particular, we'll see **homology**, **homotopy**, and **cohomology**, and we'll begin to make computations using these functors.

Preview of a proof of concept: the Brouwer fixed-point theorem

Theorem: any continuous map $f : D^2 \to D^2$ has a fixed point x (namely f(x)=x.)

[D^2 is the 2-dimensional disk with unit radius.]

'Proof': this is not really a proof, since you first have to believe what is seen by homotopy. But assuming that:



Suppose that for all x, $f(x) \neq x$. Then you can follow the line from x to f(x) backwards to a point on S^1, the boundary of D^2. This gives you a map h from $D^2 \to S^1$. You also have the inclusion $i : S^1 \subset D^2$, and when you compose the two maps $h \circ i$ [that is i, then h] you get the identity on S^1.
Here is where the magic of Alg Top comes in: apply homotopy to ends of the maps i and h, and you pass to a map between $\pi_1(S^1) \to \pi_1(S_1)$ that must be zero, since it passes through $\pi_1(D^2)$, which is trivial. But $\pi_1(S^1) = \mathbb{Z}$, and the identity map on S^1 should pass to identity on $\pi_1(S^1)$ - a contradiction.

Again, this is just a preview. We will come back to explain this better.

See the following write-up for more:
https://www.math3ma.com/blog/brouwers-fixed-point-theorem-proof

Building blocks for today: algebra

Groups

(Not fair: we didn't yet say what a group is!)
Basically, you can multiply elements *associatively*, there is a neutral element 1, and for each a there is an a^{-1} such that $a \cdot a^{-1} = 1$.
Sometimes, we use a different operation like addition (with neutral element 0) instead of multiplication.)

- Example: $F(a, b)$, the free group on two generators a and b, consists of all "words" in (positive and negative) powers of a and b.
- Problem: what group do you get if you also insist that $a \cdot b \cdot a^{-1} \cdot b^{-1} = 1$? [Answer: the abelian group on generators a and b. By multiplying the equation on the right by b and then by a, we found this is equivalent to $a \cdot b = b \cdot a$.]

Rings

Now you have two operations:

- Addition (forming a commutative group with neutral element 0)
- Multiplication - may or may not have a neutral element. Even if there is, there may not be multiplicative inverses for every element (besides 0, which never has a multiplicative inverse.) If there are (and multiplication is commutative), it's called a *field*.
- Examples: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}$. [Respectively: whole numbers (integers), rationals, reals, complex numbers, and quarternions. Confusingly, the H stands for Hamilton, their discoverer. These are formed by adding new elements i, j, and $k = i \cdot j$ to $\mathbb{R}$, such that $i^2 = j^2 = k^2 = -1$ and changing the order in multiplication can switch sign, as in: $i \cdot j = -j \cdot i$. So note that ring multiplication need not be commutative!]

Maps

Acceptable maps "morphisms" between objects of a given type must preserve the structure of the objects. For groups, a map $f : G \to H$ between groups G and H must send 1_G to 1_H, and for $g_1, g_2 \in G$, $f(g_1 \cdot_G g_2) = f(g_1) \cdot_H f(g_2)$

For rings, morphisms must 'respect' both operations.

It is often important to understand the kernel and co-kernel of a map f, illustrated in the following *exact sequence* (namely the image at one map is exactly the kernel of the next):

$0 \to \mathbb{Z} \to^{(\times 2)} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$

[A kernel of a map is the set of things the map sends to 0. A cokernel is destination 'divided by' image. In this case, the kernel of $\times 2$ is {0}, and its cokernel is $\mathbb{Z}/(2)$.]

Buiding blocks for today: topology

Spaces

- You can stretch spaces, but you can't puncture them or cut them.
- When are two spaces considered equivalent? Two possibilities:
    - Only if homeomorphic, i.e. if there is a bijection (*1-1* and *onto*, so an invertable map) between the spaces that is continuous in each direction
    - If homotopic, namely if you have maps going both ways whose compositions are homotopic to identities
- Example: P and O (ignoring serifs!) are homotopic, but not homeomorphic. Why?

Maps

Acceptable maps must respect the structure of the topological spaces considered. In most cases, one would want them to be continuous. But if one is working with a differentiable *category*, the maps would also have to be differentiable.

Building blocks for today: categories

This is a slightly newer general language useful in many areas of math. Basically a category is sort of like a 'set' on steroids [illustrate with paradox of the set of sets that don't contain themselves as an element...does it contain itself?]

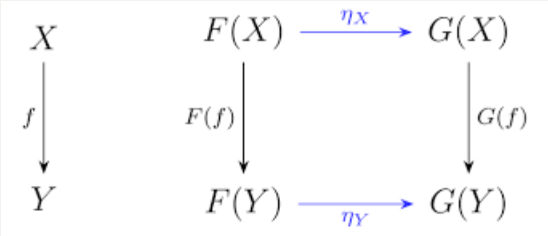
You also have 'sets' of maps or 'morphisms' between objects.

Examples:

- Topological spaces with continuous maps
- Groups with homomorphisms

Functors

Whenever you see a ("commutative") diagram like the following, good chance there's categories around:



A functor (like F or G here) is a 'map' between categories that also respects mapping between objects in the categories. For example, if f is a map between topological spaces X and Y, and F is the functor that maps into a category of groups, arrows between X and F(X) and between Y and F(Y) can be added to show the left half of the diagram is also commutative. This diagram also shows another functor G, and a way η of getting from F to G - called a 'natural transformation.' We will see exactly this between fundamental group π_1 (part of **homotopy**) and dimension 1 **homology**.

...Now we get started for real with Alg Top!

Homotopy: (graded) groups of maps of spheres into a space X.

- $\pi_n(X)$ is the group of [homotopy classes of] maps of $S^n \to X$.

[Side condition: the maps are 'based' - you think of X each S^n of having a base point, such that all maps map base point to base point. In particular that only makes sense if X is connected - if it is not, you could compute homotopy for each *component* of X.]

Problem: why is this a group? [Discussed in class - especially for π_1.]

Homology: (graded) groups formed e.g. by "chains" of simplexes of which X is composed

There are a couple ways to set up homology, ranging from the most concrete (build X out of simplexes) to definitions that are more abstract but more convenient.

Suppose X is itself a 'simplicial complex', namely composed of points, line segments, triangles, tetrahedra, hyper-tetrahedra, ... (respectively in dimensions 0,1, 2, 3, 4). Then for any n, you can take 'chains' of these, which are sums of integer multiples of n-simplexes, forming the ring C_n.

Then there is a boundary operator $\delta_n : C_n \to C_{n-1}$. It satisfies $im(\delta_{n+1}) \subset ker(\delta_n)$, so that boundary times boundary is zero: $\delta_n \circ \delta_{n+1} = 0$. (Note composition goes from right to left!)

Therefore, for each n, you can form the quotient **homology** $H_n(X) = ker(\delta_n)/im(\delta_{n+1})$

A good choice for the boundary operator is the natural geometric boundary as calculated for each simplex. For example, δ of a 2-simplex (of a triangle) is a set of three 1-simplexes (line segments.)
Careful: you have to pay attention to sign!

Problem: show that $\delta_n \circ \delta_{n+1} = 0$ in this case.
[Hint: the argument varies depending on n! We worked this out in class.]

In 'singular' homology you don't think of X as itself being made of simplexes, but rather you consider chains as composed of images of 'standard' n-dimensional hyper-tetrahedra mapped into X. This seems like a huge space, but when you take kernel 'divided' out by image, under reasonable circumstances, you get the same result. Advantages: this definition provides useful topological intuition, and you don't have to build X up as a simplicial complex.

If X is an n-dimensional manifold, you can think of k-dimensional submanifolds.

[Instead of integer multiples (our default!), you could use another 'ring of coefficients' like $\mathbb{R}$ or even $\mathbb{Z}/(2)$]

Which of these seems easier? Which seems more useful?

[Students were reluctant to offer opinions! Briefly, homotopy seems like it is easier to define, but it turns out to be much harder to compute. Where it is available, homotopy might offer deeper topological insight, but more results are avalable from homology (and especially cohomology, as we see below.)]

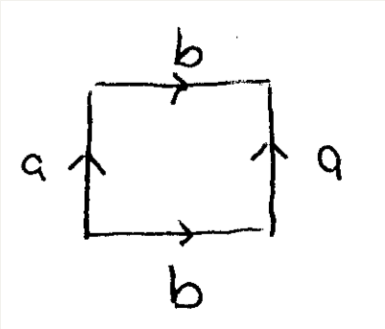
Examples

Let's illustrate both for the space S^1 (viewed as built out of three 1-simplexes, namely line segments.)
[We explained how both π_1 and H_1 are $\mathbb{Z}$, respectively counting the number of times a loop goes around the circle, and the multiple of a chain going around once (whose boundary we showed is zero.)]

Problem: what are fundamental group π_1 and 1-dimensional homology H_1 of '*' namely the space consisting only of the base point? What about the space $\mathbb{R}^3$? Can you generalize?
[Answer: these are 'contractable' spaces - namely they are homotopic to a single point. They have trivial homology (just the ring of coefficients in dimension 0) and trivial homotopy.]

Problem: what are π_1 and H_1 of $S^1 \vee S^1$, namely two circles joined at the common base point.
[Answer: $\pi_1 = F(a, b)$ (the free group on two generators), and $H_1 = \mathbb{Z} \oplus \mathbb{Z}$ (the free abelian group on two generators - much smaller than F(a, b)!)]

Problem: what are π_1 and H_1 of T^2, the 2-dimensional torus (viewed as lying flat with vertical axis of symmetry)?
[In class, we showed (with some hand waving) they are both $\mathbb{Z} \oplus \mathbb{Z}$, generated by two loops (respectively chains): 1) going around the torus from outside to inside and back, and 2) going around the torus in a horizontal circle.]
[In particular, to compute π_1, we showed we can think of the torus as the 'quotient space' of a square, where top and bottom edges are glued together, and the same for left and right edges. Then the loop going around the insides of the edges (counterclockwise from lower left) is contractable, and thus = 1 in the fundamental group. Algebraically, $b \cdot a \cdot b^{-1} \cdot a^{-1} = 1$ exactly a commutator relation. Again, an exponent of -1 indicates a backwards path]



The past three examples all illustrate the following pattern:

Theorem $H_1(X) = \pi_1(X)$ / [commutator $\pi_1(X)$], namely, $H_1(X)$ is the 'abelianization' of $\pi_1(X)$

It takes some work to prove this (which we didn't do in class.) Here are some students having trouble with this, and referring to two texts I mention above:
https://math.stackexchange.com/questions/1949774/the-first-homology-group-is-the-abelianization-of-the-fundamental-group

--> With our new experience, we can go back and take another look at the Brouwer fixed-point theorem! Note that we could prove it using either homology or homotopy.

Higher-dimensional homotopy and homology...

Theorem: $\pi_n(X)$ is commutative for $n \geq 2$.
[Proof: suggested in class. See a 'proof without words' in Totaro's article.]

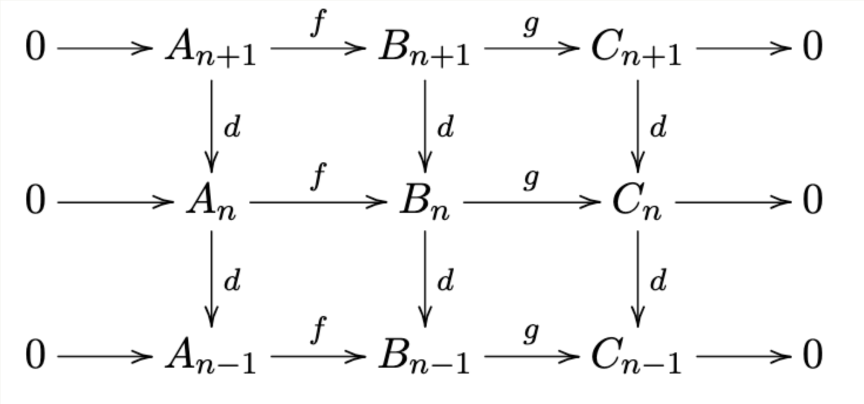
One cool thing you can do with homology: the long exact sequence

Relative homology: just 'zero out' chains within a subspace A of X

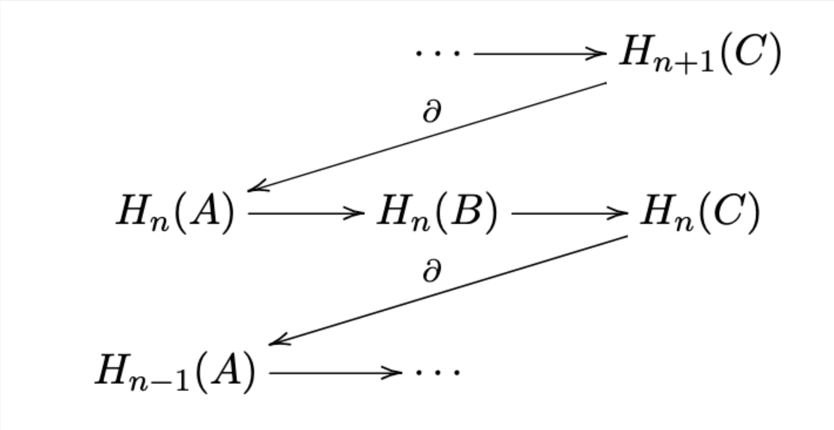
This gives you short exact sequences $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$

To make this less cluttered, we'll write this as $0 \to A_* \to B_* \to C_* \to 0$

Then you can do 'diagram chasing' on

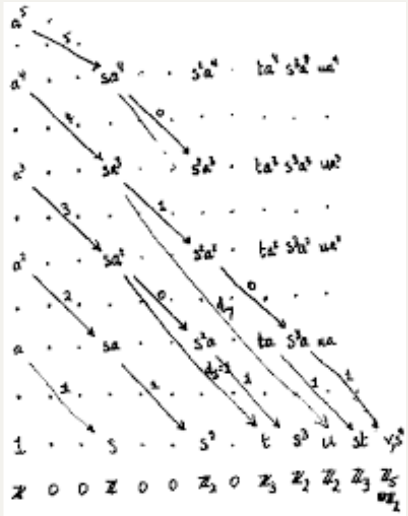


to get the following **long** exact sequence:



Knowing this is exact at each step can be a very useful help in computation!

One cool thing you can do with homotopy: spectral sequences

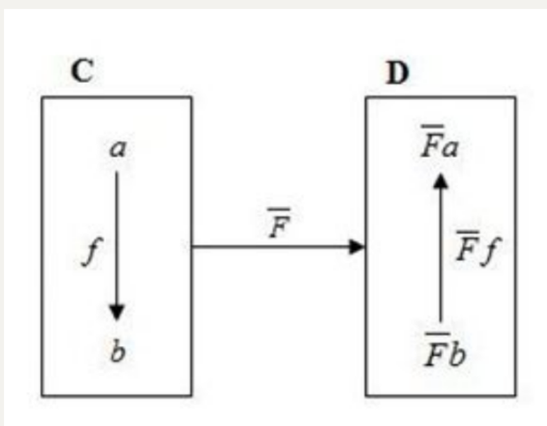


Since homotopy can be so hard to compute, complicated tools like this (which we didn't say much about) are often needed to make progress!

Cohomology: the third major functor: (graded) groups H^n formed by "co-chains" of simplexes for X

Note that n here is a superscript, not like the subscript in homology and homotopy!

Why? It's a **contravariant** functor - it makes arrows go 'backwards': a map f from a to b results in a map from cohomology of b to cohomology of a:



There are a couple different ways to do define cohomology: the earlier lower-level ways are just (acceptable) maps from chains into your ring of coefficients.

Alternately, for certain classes of spaces (e.g. reasonable n-dimensional manifolds), cohomology classes (elements) in dimension n-i correspond to homology classes in dimension i. This is called *duality*, and is another major trick facilitating calculation and interpretation.

Why would anyone go to the trouble of using this contorted, contravariant functor? Because it allows *graded* multiplication: For $g \in H^i(X), h \in H^j(X)$, there is a "cap product" $g \cap h \in H^{i+j}(X)$. This, and a couple of other tricks (like special operations between different cohomology dimensions), often make group cohomology easier to calculate for very complicated spaces!

Conclusion

You have now gained experience with two very important areas of math (modern algebra and elementary topology) that you will need very often in widely different contexts.

One very important context is when these two subjects come together in Alg Top to illuminate each other. You don't have to wait until math grad school to learn more!