# Berkeley Math Circle (Advanced): The p-adic Numbers

1. Determine the 3-adic absolute value of the following fractions
   (a) $\frac{49}{12}$
   (b) $\frac{101}{52}$
   (c) $\frac{81}{17}$

   **Solutions:**

2. Why does the sequence $\{7, 7^2, 7^3, 7^4, ...\}$ tend to zero with respect to the 7-adic absolute value?

   **Solution:**

3. (a) Add the following 5-adic numbers: $0.24242424..._5$ and $0.321321321321..._5$.
   (b) Subtract the following 3-adic numbers: $1.0202020202..._3$ and $2.101010101..._3$.
   (c) Multiply the following 2-adic numbers: $0.10101010..._2$ and $1.11111111..._2$.

   **Solutions:**

4. Express $\frac{3}{5}$ as a 2-adic decimal. You need only compute the first 4 terms.

   **Solutions:**

5. Is there a solution to $x^2 - 3 = 0$ in $\mathbb{Q}_3$? Hint: consider the potential absolute value of a solution. Hence $\mathbb{Q}_3$ does not admit a square root of 3. Does 5 admit a square root in $\mathbb{Q}_3$?

   **Solutions:**

6. Prove that if $a, b, c$ are all in $\mathbb{Q}_p$, then at least two of the numbers $\{|a-b|_p, |c-a|_p, |b-c|_p\}$ must be equal. This tells us that in $\mathbb{Q}_p$ all triangles are isosceles.

   **Solutions:**

Hensel's Lemma states the following:

Let $p$ be a prime and $f(x)$ a polynomial with integer coefficients. Then if $a$ is an integer such that $f(a) = 0\ mod(p)$ and $f'(a) \neq 0\ mod(p)$, there is a unique $p$-adic number $b$ in $\mathbb{Z}_p$ such that $f(b) = 0$ and $a = b\ mod(p)$.

7. Using Hensel's Lemma, show that 7 admits a square root in $\mathbb{Q}_3$

**Solutions:**

8. Show that $\mathbb{Q}_5$ admits a square root of $-1$, but $\mathbb{Q}_3$ does not.